A NOTE ON IDENTITIES FOR COS $N\theta$ VIA CHEBYSHEV POLYNOMIALS

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Applied mathematics uses polynomial and trigonometric functions, topics which are familiar to college freshmen. The importance of learning trigonometric identities can be reinforced by introducing the notion of Chebyshev polynomials. Also, an interesting relationship between the disciplines of algebra and trigonometry is revealed.

The first five Chebyshev polynomials are:

$$(1)\ C_0(x) = 1,\ C_1(x) = x,\quad C_2(x) = 2x^2 - 1,$$
$$C_3(x) = 4x^3 - 3x \quad \text{and} \quad C_4(x) = 8x^4 - 8x^2 + 1.$$

The pattern for finding higher degree Chebyshev polynomials involves recursion where each successive polynomial is a combination of two previous ones. Start with $C_0(x) = 1$ and $C_1(x) = x$ and use the recursive relation[1]:

$$(2)\ C_k(x) = 2xC_{k-1}(x) - C_{k-2}(x) \quad \text{for} \quad k = 2,3,4\ldots.$$

For example, we obtain $C_4(x)$ from $C_3(x)$ and $C_2(x)$ as follows

$$C_4(x) = 2xC_3(x) - C_2(x) = 2x(4x^3 - 3x) - (2x^2 - 1)$$
$$= 8x^4 - 8x^2 + 1.$$

Applications involving Chebyshev polynomials usually restrict the domain to be the interval $[-1,1]$, where the following well known property of Chebyshev polynomials is valid:

$$(3)\ C_k(x) = \cos(k \arccos(x)) \quad \text{when} \quad |x| \le 1, \text{ for all } k.$$

Using the polynomials in (1) and formula (3) and the substitution $x = \cos\theta$ gives the well known trigonometric identities:

$$(4)\ \cos 2\theta = 2\cos^2\theta - 1,$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$
$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

A general formula for $\cos k\theta$ is obtained when we substitute $x = \cos\theta$ into (2) and (3) and get a two term recurrence relation:

$$(5)\ \cos k\theta = 2\cos\theta\cos(k-1)\theta - \cos(k-2)\theta$$
$$\text{for } k = 2, 3, 4, \ldots.$$

Example. Find the formula for $\cos 5\theta$ by using the relation (5) and the identities for $\cos 3\theta$ and $\cos 4\theta$ given in (4).

Solution. $\cos 5\theta = 2\cos\theta\cos 4\theta - \cos 3\theta$
$= 2\cos\theta[8\cos^4\theta - 8\cos^2\theta + 1]$
$-4\cos^3\theta + 3\cos\theta$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

The reader may be curious how (5) can be established directly. This can be done by starting with:

(6) $\cos k\theta = \cos 2\theta \cos (k-2)\theta - \sin 2\theta \sin (k-2)\theta.$

The substitutions $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$ are used in (6) and then the terms are rearranged to yield:

(7) $\cos k\theta = 2\cos\theta[\cos\theta\cos(k-2)\theta - \sin\theta\sin(k-2)\theta] - \cos(k-2)\theta.$

An observation that the term in brackets in formula (7) is $\cos (k-1)\theta$ will complete the proof.

Remark. The functions 1, $\cos\theta$, $\cos^2\theta$ and $\cos 2\theta$ are linearly dependent, hence a formula for $\cos 2\theta$ involving a linear combination of 1, $\cos\theta$ and $\cos^2\theta$ exists. However, the functions 1, $\sin\theta$, $\sin 2\theta$ and $\sin^2\theta$ are linearly independent. This is proven by observing that their Wronskian is not identically zero[2], that is $W(\theta;1,\sin\theta,\sin 2\theta,\sin^2\theta) = 12\cos\theta \not\equiv 0$. Hence, there is no analog to formula (5) involving the set of functions $\{\sin kx\}$.

References

[1] J. H. Mathews, Numerical Methods for Computer Science, Engineering and Mathematics, Prentice-Hall, Inc., Englewood Cliffs, p. 197, [1987]

[2] A. L. Rabenstein, Elementary Differential Equations with Linear Algebra, 3rd Ed., Academic Press, New York, p. 163, [1982]