12.2 THE DIRICHLET PROBLEM FOR THE UNIT DISK

The Dirichlet problem for the unit disk $D : |z| < 1$ is to find a real-valued function $u(x, y)$ that is harmonic in the unit disk D and that takes on the boundary values

$$u(\cos\theta, \sin\theta) = U(\theta), \qquad \text{for } -\pi < \theta \leq \pi, \tag{12-10}$$

at points $z = (\cos\theta,\ \sin\theta)$ on the unit circle, as shown in Figure 12.12.

Theorem 12.7 *If $U(t)$ has period 2π and has the Fourier series representation*

$$U(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos jt + b_j \sin jt),$$

then the solution u to the Dirichlet problem in D is

$$u(r\cos\theta, r\sin\theta) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left(a_j r^j \cos j\theta + b_j r^j \sin j\theta\right), \tag{12-11}$$

where $z = x + iy = re^{i\theta}$ denotes a complex number in the closed disk $|z| \leq 1$.

The series representation in Equation (12-11) for u takes on the prescribed boundary values in Equation (12-10) at points on the unit circle $|z| = 1$. Each

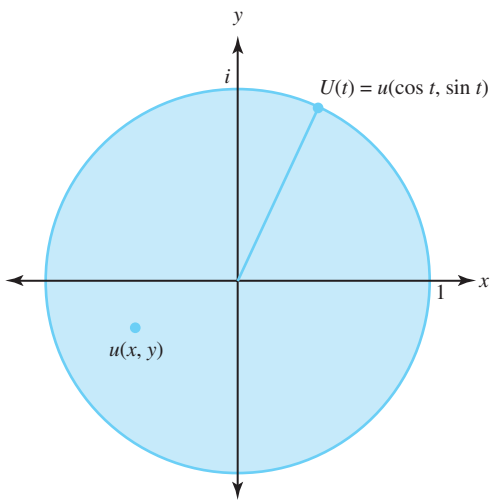

Figure 12.12 The Dirichlet problem for the unit disk $|z| < 1$.

term, $r^j \cos j\theta$ and $r^j \sin j\theta$, in the series in Equation (12-11) is harmonic, so it is reasonable to conclude that the infinite series representing u will also be harmonic. The proof follows the proof of Theorem 12.8.

Theorem 12.8 gives an integral representation for a function $u(x, y)$ that is harmonic in a domain containing the closed unit disk. The result is the analog to Poisson's integral formula for the upper half-plane.

Theorem 12.8 (Poisson integral formula for the unit disk) *Let $u(x, y)$ be a function that is harmonic in a simply connected domain that contains the closed unit disk $|z| \leq 1$. If $u(x, y)$ takes on the boundary values*

$$u(\cos\theta, \sin\theta) = U(\theta), \qquad \text{for } -\pi < \theta \leq \pi,$$

then u has the integral representation

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left(1 - r^2\right) U(t)\, dt}{1 + r^2 - 2r\cos(t - \theta)}, \tag{12-12}$$

which is valid for $|z| < 1$.

Proof Since $u(x, y)$ is harmonic in the simply connected domain, there exists a conjugate harmonic function $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is

analytic. Let C denote the contour consisting of the unit circle; then Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(\xi)\,d\xi}{\xi - z} \tag{12-13}$$

expresses the value of $f(z)$ at any point z inside C in terms of the values of $f(\xi)$ at points ξ that lie on the circle C.

If we set $z^* = (\overline{z})^{-1}$, then z^* lies outside the unit circle C and the Cauchy–Goursat theorem establishes the equation

$$0 = \frac{1}{2\pi i}\int_c \frac{f(\xi)\,d\xi}{\xi - z^*}. \tag{12-14}$$

Subtracting Equation (12-14) from Equation (12-13) and using the parameterization $\xi = e^{it}$, $d\xi = ie^{it}\,dt$ and the substitutions $z = re^{i\theta}$, $z^* = \frac{1}{r}e^{i\theta}$ gives

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{it}}{e^{it} - re^{i\theta}} - \frac{e^{it}}{e^{it} - \frac{1}{r}e^{i\theta}}\right) f\left(e^{it}\right) dt.$$

We rewrite the expression inside the parentheses on the right side of this equation as

$$\begin{aligned}\frac{e^{it}}{e^{it} - re^{i\theta}} - \frac{e^{it}}{e^{it} - \frac{1}{r}e^{i\theta}} &= \frac{1}{1 - re^{i(\theta - t)}} + \frac{re^{i(t-\theta)}}{1 - re^{i(t-\theta)}} \\ &= \frac{1 - r^2}{1 + r^2 - 2r\cos(t - \theta)},\end{aligned} \tag{12-15}$$

and it follows that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left(1 - r^2\right) f\left(e^{it}\right) dt}{1 + r^2 - 2r\cos(t - \theta)}.$$

Because $u(x, y)$ is the real part of $f(z)$ and $U(t)$ is the real part of $f\left(e^{it}\right)$, we can equate the real parts in the preceding equation to obtain Equation (12-12), completing the proof of Theorem 12.8.

We now turn to the proof Theorem 12.7. The real-valued function

$$P(r, t - \theta) = \frac{1 - r^2}{1 + r^2 - 2r\cos(t - \theta)}$$

is known as the **Poisson kernel**. Expanding the left side of Equation (12-15) in a geometric series gives

$$\begin{aligned}
P(r, t-\theta) &= \frac{1}{1-re^{i(\theta-t)}} + \frac{re^{i(t-\theta)}}{1-re^{i(t-\theta)}} = \sum_{n=0}^{\infty} r^n e^{in(\theta-t)} + \sum_{n=1}^{\infty} r^n e^{in(t-\theta)} \\
&= 1 + \sum_{n=1}^{\infty} r^n \left[e^{in(\theta-t)} + e^{in(t-\theta)}\right] = 1 + 2\sum_{n=1}^{\infty} r^n \cos\left[n(\theta-t)\right] \\
&= 1 + 2\sum_{n=1}^{\infty} r^n \left(\cos n\theta \cos nt + \sin n\theta \sin nt\right) \\
&= 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta \cos nt + 2\sum_{n=1}^{\infty} r^n \sin n\theta \sin nt.
\end{aligned}$$

We now use this result in Equation (12-12) to obtain

$$\begin{aligned}
u(r\cos\theta, r\sin\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, t-\theta)\, U(t)\, dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(t)\, dt + \frac{1}{\pi}\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} r^n \cos n\theta \cos nt\ U(t)\ dt \\
&\quad + \frac{1}{\pi}\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} r^n \sin n\theta \cos nt\ U(t)\ dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(t)\ dt + \sum_{n=1}^{\infty} \frac{r^n}{\pi} \cos n\theta \int_{-\pi}^{\pi} \cos nt\ U(t)\ dt \\
&\quad + \sum_{n=1}^{\infty} \frac{r^n}{\pi} \sin n\theta \int_{-\pi}^{\pi} \sin nt\ U(t)\ dt \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + \sum_{n=1}^{\infty} b_n r^n \sin n\theta,
\end{aligned}$$

where $\{a_n\}$ and $\{b_n\}$ are the Fourier series coefficients for $U(t)$. This result establishes the representation for $u(r\cos\theta, r\sin\theta)$ in Equation (12-11) of Theorem 12.7.