12.4 THE FOURIER TRANSFORM

If we let $U(t)$ be a real-valued function with period 2π, which is piecewise continuous such that $U'(t)$ also exists and is piecewise continuous, then $U(t)$ has the **complex Fourier series** representation

$$U(t) = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(t)\, e^{-int} dt, \qquad \text{for all } n.$$

The coefficients $\{c_n\}$ are complex numbers. Previously, we expressed $U(t)$ as the real trigonometric series

$$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{12-19}$$

Hence a relationship between the coefficients is

$$\begin{aligned} a_n &= c_n + c_{-n}, &\quad& \text{for } n = 0, 1, ..., \quad \text{and} \\ b_n &= i(c_n - c_{-n}), && \text{for } n = 1, 2, ... \end{aligned}$$

We can easily establish these relations. We start by writing

$$\begin{aligned} U(t) &= c_0 + \sum_{n=1}^{\infty} c_n e^{int} + \sum_{n=1}^{\infty} c_{-n} e^{-int} \qquad (12\text{-}20) \\ &= c_0 + \sum_{n=1}^{\infty} c_n (\cos nt + i \sin nt) + \sum_{n=1}^{\infty} c_{-n} (\cos nt - i \sin nt) \\ &= c_0 + \sum_{n=1}^{\infty} [(c_n + c_{-n}) \cos nt + i (c_n - c_{-n}) \sin nt]. \end{aligned}$$

Comparing Equations (12-20) and (12-19), we see that $a_0 = 2c_0$, $a_n = c_n + c_{-n}$, and $b_n = i(c_n - c_{-n})$.

If $U(t)$ and $U'(t)$ are piecewise continuous and have period $2L$, then $U(t)$ has the complex Fourier series representation

$$U(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi nt/L}, \tag{12-21}$$

where

$$c_n = \frac{1}{2L} \int_{-L}^{L} U(t)\, e^{-i\pi nt/L} dt, \qquad \text{for all } n. \tag{12-22}$$

We've shown how periodic functions are represented by trigonometric series, but many practical problems involve nonperiodic functions. A representation analogous to a Fourier series for a nonperiodic function $U(t)$ is obtained by considering the Fourier series of $U(t)$ for $-L < t < L$ and then taking the limit as $L \to \infty$. The result is known as the **Fourier transform** of $U(t)$.

We start with the nonperiodic function $U(t)$ and consider the periodic function $U_L(t)$ with period $2L$, where

$$\begin{array}{ll} U_L(t) = U(t), & \text{for } -L < t \le L, \quad \text{and} \\ U_L(t) = U_L(t + 2L), & \text{for all } t. \end{array}$$

Then $U_L(t)$ has the complex Fourier series representation

$$U_L(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi nt/L}. \tag{12-23}$$

We need to introduce some terminology in order to discuss the terms in Equation (12-23). First

$$w_n = \frac{\pi n}{L} \tag{12-24}$$

is called the **frequency**. If t denotes time, then the units for w_n are radians per unit time. The set of all possible frequencies is called the **frequency spectrum**, that is,

$$\left\{ \ldots, \frac{-3\pi}{L}, \frac{-2\pi}{L}, \frac{-\pi}{L}, \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots \right\}.$$

Note that, as L increases, the spectrum becomes finer and approaches a continuous spectrum of frequencies. It is reasonable to expect that the summation in the Fourier series for $U_L(t)$ will give rise to an integral over $[-\infty,\infty]$. This result is stated in Theorem 12.9.

Theorem 12.9 (Fourier transform) *Let $U(t)$ and $U'(t)$ be piecewise continuous and*

$$\int_{-\infty}^{\infty} |U(t)|\, dt < M$$

for some positive constant M. The Fourier transform $F(w)$ of $U(t)$ is defined as

$$F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(t)\, e^{-iwt} dt. \tag{12-25}$$

At points of continuity, $U(t)$ has the integral representation

$$U(t) = \int_{-\infty}^{\infty} F(w)\, e^{iwt} dw,$$

and at a point $t = a$ of discontinuity of U, the integral converges to $\dfrac{U(a^-) + U(a^+)}{2}$.

The fact that U is transformed into F is commonly expressed by the operator notation

$$\mathfrak{F}(U(t)) = F(w).$$

Proof Set $\Delta w_n = w_{n+1} - w_n = \dfrac{\pi}{L}$ and $\dfrac{1}{2L} = \dfrac{1}{2\pi}\Delta w_n$. These quantities are used in conjunction with Equations (12-21), (12-22), and (12-23) and the frequency in Equation (12-24) to obtain

$$\begin{aligned} U_L(t) &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L} \int_{-L}^{L} U(t)\, e^{-iw_n t} dt \right] e^{iw_n t} \qquad (12\text{-}26) \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-L}^{L} U(t)\, e^{-iw_n t} dt \right] e^{iw_n t} \Delta w_n. \end{aligned}$$

If we define $F_L(w)$ by

$$F_L(w) = \frac{1}{2\pi} \int_{-L}^{L} U(t)\, e^{-iwt} dt,$$

then we can write Equation (12-26) as

$$U_L(t) = \sum_{n=-\infty}^{\infty} F_L(w_n)\, e^{iw_n t} \Delta w_n. \tag{12-27}$$

As L gets large, $F_L(w_n)$ approaches $F(w_n)$ and Δw_n tends to zero. Thus the limit on the right side of Equation (12-27) can be viewed as an integral, which substantiates the Fourier integral representation

$$U(t) = \int_{-\infty}^{\infty} F(w)\, e^{iwt} dw.$$

A more rigorous proof of this fact is presented in various advanced texts.

Table 12.1 gives some important properties of the Fourier transform.

Linearity	$\mathfrak{F}(aU_1(t) + bU_2(t)) = a\,\mathfrak{F}(U_1(t)) + b\,\mathfrak{F}(U_2(t))$
Symmetry	If $\mathfrak{F}(U(t)) = F(w)$, then $\mathfrak{F}(F(t)) = \frac{1}{2\pi}U(-w)$.
Time scaling	$\mathfrak{F}(U(at)) = \frac{1}{\lvert a\rvert}F\left(\frac{w}{a}\right)$
Time shifting	$\mathfrak{F}(U(t-t_0)) = e^{-it_0w}F(w)$
Frequency shifting	$\mathfrak{F}\left(e^{iw_0t}U(t)\right) = F(w-w_0)$
Time differentiation	$\mathfrak{F}(U'(t)) = iwF(w)$
Frequency differentiation	$\frac{d^nF(w)}{dw^n} = \mathfrak{F}\left((-it)^n U(t)\right)$
Moment theorem	If $M_n = \int_{-\infty}^{\infty} t^n U(t)\, dt$, then $(-i)^n M_n = 2\pi F^{(n)}(0)$.

Table 12.1 Properties of the Fourier Transform