

► **Theorem 12.24 (Convolution theorem)** Let $F(s)$ and $G(s)$ denote the Laplace transforms of $f(t)$ and $g(t)$, respectively. Then the product given by $H(s) = F(s)G(s)$ is the Laplace transform of the convolution of f and g , is denoted $h(t) = (f * g)(t)$, and has the integral representation

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau \quad \text{or} \quad (12-45)$$

$$h(t) = (g * f)(t) = \int_0^t g(\tau)f(t - \tau) d\tau. \quad (12-46)$$

Proof The following proof is given for the special case when s is a real number. The general case is covered in advanced texts. Using the dummy variables σ and τ and the integrals defining the transforms, we can express their product as

$$F(s)G(s) = \left[\int_0^\infty f(\sigma)e^{-s\sigma} d\sigma \right] \left[\int_0^\infty g(\tau)e^{-s\tau} d\tau \right].$$

The product of integrals in this equation can be written as an iterated integral:

$$F(s)G(s) = \int_0^\infty \left[\int_0^\infty f(\sigma)e^{-s(\sigma+\tau)} d\sigma \right] g(\tau) d\tau.$$

We hold τ fixed, use the change of variables $t = \sigma + \tau$ and $dt = d\sigma$, and rewrite the inner integral in the equation to obtain

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \left[\int_\tau^\infty f(t - \tau)e^{-st} dt \right] g(\tau) d\tau \\ &= \int_0^\infty \left[\int_\tau^\infty f(t - \tau)g(\tau)e^{-st} dt \right] d\tau. \end{aligned}$$

The region of integration for this last iterated integral is the wedge-shaped region in the (t, τ) plane shown in Figure 12.28. We reverse the order of integration in the integral to get

$$F(s)G(s) = \int_0^\infty \left[\int_0^t f(t - \tau)g(\tau)e^{-st} d\tau \right] dt.$$

We rewrite this equation as

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \left[\int_0^t f(t - \tau)g(\tau) d\tau \right] e^{-st} dt \\ &= \mathcal{L}^{-1} \left(\int_0^t f(t - \tau)g(\tau) d\tau \right), \end{aligned}$$

which establishes Equation (12-46). We can interchange the role of the functions $f(t)$ and $g(t)$, so Equation (12-45) follows immediately.

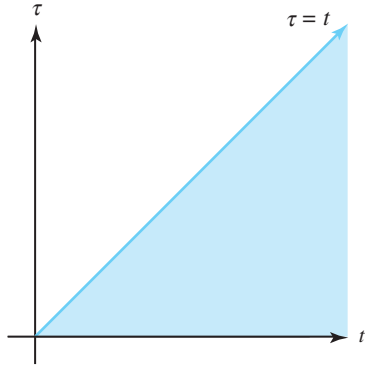


Figure 12.28 The region of integration in the convolution theorem.

Table 12.4 lists the properties of convolution.

Commutative	$f * g = g * f$
Distributive	$f * (g + h) = f * g + f * h$
Associative	$(f * g) * h = f * (g * h)$
Zero	$f * 0 = 0$

Table 12.4 Properties of Convolution