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The convolution method can be used to solve initial value problems. The tedious mechanical details of problem solving can be facilitated with computer software such as Maple$^{\text{TM}}$, Matlab$^{\text{TM}}$, or Mathematica$^{\text{TM}}$.

Theorem 12.25 (Initial value problem (IVP) convolution method) *The unique solution to the initial value problem*

$$ay''(t) + by'(t) + cy(t) = g(t) \text{ with } y(0) = y_0 \text{ and } y'(0) = y_1$$

is given by

$$y(t) = u(t) + (h * g)(t),$$

where $u(t)$ *is the solution to the homogeneous equation*

$$au''(t) + bu'(t) + cu(t) = 0 \text{ with } u(0) = y_0 \text{ and } u'(0) = y_1,$$

and $h(t)$ has the Laplace transform given by $H(s) = \dfrac{1}{as^2 + bs + c}$.

Proof The particular solution is found by solving the equation

$$av''(t) + bv'(t) + cv(t) = g(t), \qquad \text{with } v(0) = 0 \text{ and } v'(0) = 0.$$

Taking the Laplace transform of both sides of this equation produces

$$as^2V(s) + bsV(s) + cV(s) = G(s).$$

Solving for $V(s)$ in this equation yields $V(s) = \dfrac{1}{as^2 + bs + c}G(s)$. If we set $H(s) = \dfrac{1}{as^2 + bs + c}$, then $V(s) = H(s)G(s)$ and the particular solution is given by the convolution

$$v(t) = (h * g)(t).$$

The general solution is $y(t) = u(t) + v(t) = u(t) + (h * g)(t)$. To verify that the initial conditions are met, we compute

$$\begin{aligned} y(0) &= u(0) + v(0) = y_0 + 0 = y_0 \quad \text{and} \\ y'(0) &= u'(0) + v'(0) = y_1 + 0 = y_1, \end{aligned}$$

completing the proof of the theorem.