12.6 LAPLACE TRANSFORMS OF DERIVATIVES AND INTEGRALS

Theorem 12.13 (Differentiation of $f(t)$) *Let $f(t)$ and $f'(t)$ be continuous for $t \geq 0$ and be of exponential order. Then,*

$$\mathcal{L}(f'(t)) = sF(s) - f(0),$$

where

$$F(s) = \mathcal{L}(f(t)).$$

Proof Let K be large enough that both $f(t)$ and $f'(t)$ are of exponential order K. If $\operatorname{Re}(s) > K$, then $\mathcal{L}(f'(t))$ is given by

$$\mathcal{L}(f'(t)) = \int_0^\infty f'(t)\, e^{-st} dt.$$

Next, using integration by parts, we rewrite this equation as

$$\mathcal{L}\left(f'(t)\right) = \lim_{R\to+\infty}\left[f(t)\,e^{-st}\right]\Big|_{t=0}^{t=R} + s\int_0^{\infty} f(t)\,e^{-st}dt.$$

As $f(t)$ is of exponential order K and $\operatorname{Re}(s) > K$, we have $\lim_{R\to+\infty} f(R)\,e^{-sR} = 0$. Hence the preceding equation becomes

$$\mathcal{L}\left(f'(t)\right) = -f(0) + s\int_0^{\infty} f(t)\,e^{-st}dt = sF(s) - f(0)\,,$$

proving the theorem.

Corollary 12.1 *If $f(t)$, $f'(t)$, and $f''(t)$ are of exponential order, then*

$$\mathcal{L}\left(f''(t)\right) = s^2 f(s) - sf(0) - f'(0)\,.$$

■