Theorem 12.14 (Integration of $f(t)$) *Let $f(t)$ be continuous for $t \geq 0$ and of exponential order and let $F(s)$ be its Laplace transform. Then*

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}.$$

Proof Let $g(t) = \int_0^t f(\tau)\,d\tau$. Then, $g'(t) = f(t)$ and $g(0) = 0$. If we can show that g is of exponential order, then Theorem 12.13 will imply that

$$\mathcal{L}(f(t)) = \mathcal{L}(g'(t)) = s\mathcal{L}(g(t)) - 0 = s\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right),$$

and the proof will be complete. As $f(t)$ is of exponential order, we can find positive values M and K so that

$$|g(t)| \leq \int_0^t f(\tau)\, d\tau \leq M \int_0^t e^{K\tau} d\tau = \frac{M}{K}\left(e^{Kt} - 1\right) \leq e^{Kt},$$

establishing that g is of exponential order and completing the proof.