12.8 MULTIPLICATION AND DIVISION BY t

Sometimes the solutions to nonhomogeneous linear differential equations with constant coefficients involve the functions $t\cos bt$, $t\sin bt$, or $t^n e^{at}$ as part of the solution. We now show how the Laplace transforms of $tf(t)$ and $\frac{f(t)}{t}$ are related to the Laplace transform of $f(t)$. We obtain the transform of $tf(t)$ via differentiation and the transform of $\frac{f(t)}{t}$ via integration. To be precise, we present Theorems 12.17 and 12.18.

Theorem 12.17 (Multiplication by t) *If $F(s)$ is the Laplace transform of $f(t)$, then*

$$\mathcal{L}(tf(t)) = -F'(s).$$

Proof By definition, we have $F(s) = \int_0^\infty f(t)\,e^{-st}dt$. Leibniz's rule (Theorem 6.11) for partial differentiation under the integral sign permits us to write

$$\begin{aligned} F'(s) &= \frac{\partial}{\partial s}\int_0^\infty f(t)\,e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s}\left[f(t)\,e^{-st}\right]dt \\ &= \int_0^\infty \left[-tf(t)\,e^{-st}\right]dt = -\int_0^\infty tf(t)\,e^{-st}dt \\ &= -\mathcal{L}(tf(t)), \end{aligned}$$

establishing the result.