12.8 MULTIPLICATION AND DIVISION BY t

Sometimes the solutions to nonhomogeneous linear differential equations with constant coefficients involve the functions $t\cos bt$, $t\sin bt$, or $t^n e^{at}$ as part of the solution. We now show how the Laplace transforms of $tf(t)$ and $\frac{f(t)}{t}$ are related to the Laplace transform of $f(t)$. We obtain the transform of $tf(t)$ via differentiation and the transform of $\frac{f(t)}{t}$ via integration. To be precise, we present Theorems 12.17 and 12.18.

Theorem 12.18 (Division by t) *Let both $f(t)$ and $\frac{f(t)}{t}$ have Laplace transforms and let $F(s)$ denote the transform of $f(t)$. If $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists, then*

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)\, d\sigma.$$

Proof Because $F(\sigma) = \int_0^\infty f(t)\, e^{-\sigma t} dt$, we integrate $F(\sigma)$ from s to ∞ and obtain

$$\int_s^\infty F(\sigma)\, d\sigma = \int_0^\infty \left[\int_0^\infty f(t)\, e^{-\sigma t} dt\right] d\sigma.$$

We reverse the order of integration in the double integral of this equation to obtain

$$\begin{aligned}\int_s^\infty F(\sigma)\,d\sigma &= \int_0^\infty \left[\int_s^\infty f(t)\,e^{-\sigma t}d\sigma\right] dt \\ &= \int_0^\infty \left[\frac{-f(t)}{t}e^{-\sigma t}\Bigg|_{\sigma=s}^{\sigma=\infty}\right] dt \\ &= \int_0^\infty \frac{f(t)}{t}e^{-st}dt = \mathcal{L}\left(\frac{f(t)}{t}\right),\end{aligned}$$

completing the proof.