Definition 12.3: Unit step function

Let $a \geq 0$. Then, the **unit step function** $U_a(t)$ is

$$U_a(t) = \begin{cases} 0, & \textit{for } t < a; \\ 1, & \textit{for } t > a. \end{cases}$$

The graph of $U_a(t)$ is shown in Figure 12.22.

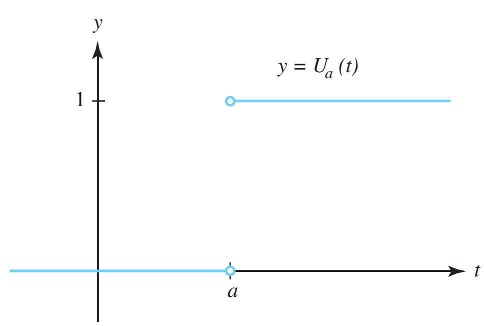

Figure 12.22 The graph of the unit step function $y = U_a(t)$.

Theorem 12.16 (Shifting the variable t) *If $F(s)$ is the Laplace transform of $f(t)$ and $a \geq 0$, then*

$$\mathcal{L}(U_a(t)\, f(t-a)) = e^{-as} F(s),$$

where $f(t)$ and $U_a(t)(t-a)$ are illustrated in Figure 12.23.

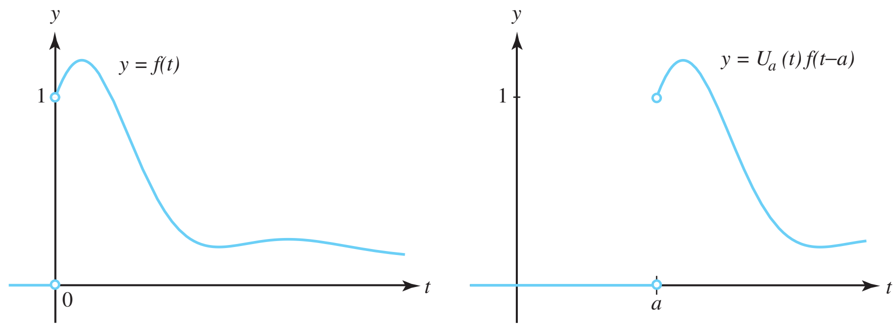

Figure 12.23 Comparison of the functions $f(t)$ and $U_a(t)\ f(t-a)$.

Proof Using the definition of the Laplace transform, we write

$$e^{-as}F(s) = e^{-as}\int_0^\infty f(\tau)\,e^{-s\tau}d\tau = \int_0^\infty f(\tau)\,e^{-s(a+\tau)}d\tau.$$

Using the change of variable $t = a + \tau$ and $dt = d\tau$, we obtain

$$e^{-as}F(s) = \int_a^\infty f(t-a)\,e^{-st}dt.$$

Because $U_a(t)\ f(t-a) = 0$, for $t < a$, and $U_a(t)\ f(t-a) = f(t-a)$, for $t > a$, we rewrite the preceding equation as

$$e^{-as}F(s) = \int_0^\infty U_a(t)\ f(t-a)\,e^{-st}dt = \mathcal{L}\left(U_a(t)\ f(t-a)\right),$$

and the proof is complete.