12.5 THE LAPLACE TRANSFORM

In this section we investigate a very powerful tool for engineering applications.

12.5.1 From the Fourier Transform to the Laplace Transform

We have shown that certain real-valued functions $f(t)$ have a Fourier transform and that the integral

$$g(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

defines the complex function $g(\omega)$ of the real variable ω. If we multiply the integrand $f(t)\, e^{-i\omega t}$ by $e^{-\sigma t}$, then we create a complex function $G(\sigma + i\omega)$ of the complex variable $\sigma + i\omega$:

$$G(\sigma + i\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-\sigma t} e^{-i\omega t} dt = \int_{-\infty}^{\infty} f(t)\, e^{-(\sigma + i\omega)t} dt.$$

The function $G(\sigma + i\omega)$ is called the **two-sided Laplace transform** of $f(t)$, and it exists when the Fourier transform of the function $f(t)\, e^{-\sigma t}$ exists. From Fourier transform theory, a sufficient condition for $G(\sigma + iw)$ to exist is that

$$\int_{-\infty}^{\infty} |f(t)|\, e^{-\sigma t} dt < \infty.$$

For a function $f(t)$, this integral is finite for values of σ that lie in some interval $a < \sigma < b$.

The two-sided Laplace transform has the lower limit of integration $t = -\infty$ and hence requires a knowledge of the past history of the function $f(t)$ (i.e., when $t < 0$). For most physical applications, we are interested in the behavior of a system only for $t \geq 0$. The initial conditions $f(0)$, $f'(0)$, $f''(0)$, ... are a consequence of the past history of the system and are often all that we know. For this reason, it is useful to define the one-sided Laplace transform of $f(t)$,

which is commonly referred to simply as the **Laplace transform** of $f(t)$, which is also defined as an integral:

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)\, e^{-st} dt, \tag{12-28}$$

where $s = \sigma + i\omega$. If the integral in Equation (12-28) for the Laplace transform exists for $s_0 = \sigma_0 + i\omega$, then values of σ with $\sigma > \sigma_0$ imply that $e^{-\sigma t} < e^{-\sigma_0 t}$ and so

$$\int_0^\infty |f(t)|\, e^{-\sigma t} dt < \int_0^\infty |f(t)|\, e^{-\sigma_0 t} dt < \infty,$$

from which it follows that $F(s)$ exists for $s = \sigma + i\omega$. Therefore, the Laplace transform $\mathcal{L}(f(t))$ is defined for all points s in the right half-plane $\mathrm{Re}(s) > \sigma_0$.

Another way to view the relationship between the Fourier transform and the Laplace transform is to consider the function $U(t)$ given by

$$U(t) = \begin{cases} f(t), & \text{for } t \geq 0; \\ 0, & \text{for } t < 0. \end{cases}$$

Then the Fourier transform theory shows that

$$U(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} U(t)\, e^{-i\omega t} dt \right] e^{i\omega t} d\omega,$$

and, because the integrand $U(t)$ is zero for $t < 0$, we can write this equation as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{0}^{\infty} f(t)\, e^{-i\omega t} dt \right] e^{i\omega t} d\omega.$$

If we use the change of variable $s = \sigma + i\omega$ and $d\omega = (ds/i)$, holding $\sigma > \sigma_0$ fixed, then the new limits of integration are from $s = \sigma - i\omega$ to $s = \sigma + i\omega$. The resulting equation is

$$f(t) = \frac{1}{2\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} \left[\int_{0}^{\infty} f(t)\, e^{-st} dt \right] e^{st} ds.$$

Therefore, the Laplace transform is

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)\, e^{-st} dt, \text{ where } s = \sigma + i\omega,$$

and the inverse Laplace transform is

$$\mathcal{L}^{-1}(F(s)) = f(t) = \frac{1}{2\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\, e^{st} ds. \tag{12-29}$$

12.5.2 Properties of the Laplace Transform

Although a function $f(t)$ may be defined for all values of t, its Laplace transform is not influenced by values of $f(t)$, where $t < 0$. The Laplace transform of $f(t)$ is actually defined for the function $U(t)$ given in the last section by

$$U(t) = \begin{cases} f(t), & \text{for } t \geq 0; \\ 0, & \text{for } t < 0. \end{cases}$$

A sufficient condition for the existence of the Laplace transform is that $|f(t)|$ not grow too rapidly as $t \to +\infty$. We say that the function f is of *exponential order* if there exist real constants $M > 0$ and K, such that

$$|f(t)| \leq Me^{Kt}$$

holds for all $t \geq 0$. All functions in this chapter are assumed to be of exponential order. Theorem 12.10 shows that the Laplace transform $F(\sigma + i\tau)$ exists for values of s in a domain that includes the right half-plane $\operatorname{Re}(s) > K$.

Theorem 12.10 (Existence of the Laplace transform) *If f is of exponential order, then its Laplace transform $\mathcal{L}(f(t)) = F(s)$ is given by*

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt,$$

where $s = \sigma + i\omega$. The defining integral for F exists at points $s = \sigma + i\tau$ in the right half-plane $\sigma > K$.

Proof Using $s = \sigma + i\tau$, we can express $F(s)$ as

$$F(s) = \int_0^\infty f(t)\, e^{-\sigma t} \cos \tau t \; dt - i \int_0^\infty f(t)\, e^{-\sigma t} \sin \tau t \; dt.$$

Then for values of $\sigma > K$, we have

$$\int_0^\infty |f(t)|\, e^{-\sigma t} \, |\cos \tau t| \, dt \leq M \int_0^\infty e^{(K-\sigma)t} dt \leq \frac{M}{\sigma - K} \quad \text{and}$$

$$\int_0^\infty |f(t)|\, e^{-\sigma t} \, |\sin \tau t| \, dt \leq M \int_0^\infty e^{(K-\sigma)t} dt \leq \frac{M}{\sigma - K},$$

which imply that the integrals defining the real and imaginary parts of F exist for values of $\operatorname{Re}(s) > K$, completing the proof.

Remark 12.1 The domain of definition of the defining integral for the Laplace transform $\mathcal{L}(f(t))$ seems to be restricted to a half-plane. However, the resulting

formula $F(s)$ might have a domain much larger than this half-plane. Later we show that $F(s)$ is an analytic function of the complex variable s. For most applications involving Laplace transforms that we present, the Laplace transforms are rational functions that take the form $\dfrac{P(s)}{Q(s)}$, where P and Q are polynomials; in other important applications, the functions take the form $\dfrac{e^{as}P(s)}{Q(s)}$. ■

Theorem 12.11 (Linearity of the Laplace transform) *Let f and g have Laplace transforms F and G, respectively. If a and b are constants, then*

$$\mathcal{L}(af(t)+bg(t)) = aF(s)+bG(s).$$

Proof Let K be chosen so that both F and G are defined for $\operatorname{Re}(s) > K$. Then

$$\begin{aligned}\mathcal{L}(af(t)+bg(t)) &= \int_0^\infty [af(t)+bg(t)]\,e^{-st}dt \\ &= a\int_0^\infty f(t)\,e^{-st}dt + b\int_0^\infty g(t)\,e^{-st}dt \\ &= aF(s)+bG(s).\end{aligned}$$

Theorem 12.12 (Uniqueness of the Laplace transform) *Let f and g have Laplace transforms, F and G, respectively. If $F(s) \equiv G(s)$, then $f(t) \equiv g(t)$.*

Proof If σ is sufficiently large, then the integral representation, Equation (12-29), for the inverse Laplace transform can be used to obtain

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\,e^{st}ds = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} G(s)\,e^{st}ds \\ &= \mathcal{L}^{-1}(G(s)) = g(t),\end{aligned}$$

and the theorem is proven.