◗ Theorem 6.14 (Gauss's mean value theorem) *If f is analytic in a simply connected domain D that contains the circle $C_R(z_0) = \{z : |z - z_0| = R\}$, then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + Re^{i\theta}\right) d\theta.$$

Proof We parametrize the circle $C_R(z_0)$ by

$$C_R(z_0) : z(\theta) = z_0 + Re^{i\theta} \quad \text{and} \quad dz = iRe^{i\theta} d\theta, \qquad \text{for } 0 \le \theta \le 2\pi,$$

and use this parametrization along with Cauchy's integral formula to obtain

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(z_0 + Re^{i\theta}\right) iRe^{i\theta}\, d\theta}{Re^{i\theta}} = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + Re^{i\theta}\right) d\theta.$$