Back Substitution

3.3 Upper-Triangular Linear Systems

We will now develop the ***back-substitution algorithm***, which is useful for solving a linear system of equations that has an upper-triangular coefficient matrix. This algorithm will be incorporated in the algorithm for solving a general linear system in Section 3.4.

Definition 3.2. An $N \times N$ matrix $\boldsymbol{A} = [a_{ij}]$ is called ***upper triangular*** provided that the elements satisfy $a_{ij} = 0$ whenever $i > j$. The $N \times N$ matrix $\boldsymbol{A} = [a_{ij}]$ is called ***lower triangular*** provided that $a_{ij} = 0$ whenever $i < j$. ▲

We will develop a method for constructing the solution to upper-triangular linear systems of equations and leave the investigation of lower-triangular systems to the reader. If $\boldsymbol{A}$ is an upper-triangular matrix, then $\boldsymbol{AX} = \boldsymbol{B}$ is said to be an ***upper-***

triangular system of linear equations and has the form

$$
\begin{array}{rrrrl}
a_{11}x_1+a_{12}x_2+ & a_{13}x_3+\cdots+ & a_{1N-1}x_{N-1}+ & a_{1N}x_N & = b_1 \\
a_{22}x_2+ & a_{23}x_3+\cdots+ & a_{2N-1}x_{N-1}+ & a_{2N}x_N & = b_2 \\
 & a_{33}x_3+\cdots+ & a_{3N-1}x_{N-1}+ & a_{3N}x_N & = b_3 \\
 & & \vdots & & \vdots \\
 & & a_{N-1N-1}x_{N-1}+ & a_{N-1N}x_N & = b_{N-1} \\
 & & & a_{NN}x_N & = b_N.
\end{array} \tag{1}
$$

Theorem 3.5 (Back Substitution). Suppose that $\boldsymbol{AX} = \boldsymbol{B}$ is an upper-triangular system with the form given in (1). If

$$a_{kk} \neq 0 \quad \text{for } k = 1, 2, \ldots, N, \tag{2}$$

then there exists a unique solution to (1).

Constructive Proof. The solution is easy to find. The last equation involves only x_N, so we solve it first:

$$x_N = \frac{b_N}{a_{NN}}. \tag{3}$$

Now x_N is known and it can be used in the next-to-last equation:

$$x_{N-1} = \frac{b_{N-1} - a_{N-1N}x_N}{a_{N-1N-1}}. \tag{4}$$

Now x_N and x_{N-1} are used to find x_{N-2}:

$$x_{N-2} = \frac{b_{N-2} - a_{N-2N-1}x_{N-1} - a_{N-2N}x_N}{a_{N-2N-2}}. \tag{5}$$

Once the values $x_N, x_{N-1}, \ldots, x_{k+1}$ are known, the general step is

$$x_k = \frac{b_k - \sum_{j=k+1}^{N} a_{kj}x_j}{a_{kk}} \quad \text{for } k = N-1, N-2, \ldots, 1. \tag{6}$$

The uniqueness of the solution is easy to see. The Nth equation implies that b_N/a_{NN} is the only possible value of x_N. Then finite induction is used to establish that $x_{N-1}, x_{N-2}, \ldots, x_1$ are unique. •

Example 3.12. Use back substitution to solve the linear system

$$
\begin{aligned}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
-2x_2 + 7x_3 - 4x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6.
\end{aligned}
$$

Solving for x_4 in the last equation yields

$$x_4 = \frac{6}{3} = 2.$$

Using $x_4 = 2$ in the third equation, we obtain

$$x_3 = \frac{4 - 5(2)}{6} = -1.$$

Now $x_3 = -1$ and $x_4 = 2$ are used to find x_2 in the second equation:

$$x_2 = \frac{-7 - 7(-1) + 4(2)}{-2} = -4.$$

Finally, x_1 is obtained using the first equation:

$$x_1 = \frac{20 + 1(-4) - 2(-1) - 3(2)}{4} = 3. \qquad \blacksquare$$

The condition that $a_{kk} \neq 0$ is essential because equation (6) involves division by a_{kk}. If this requirement is not fulfilled, either no solution exists or infinitely many solutions exist.

Example 3.13. Show that there is no solution to the linear system

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ 0x_2 + 7x_3 - 4x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6. \end{aligned} \tag{7}$$

Using the last equation in (7), we must have $x_4 = 2$, which is substituted into the second and third equations to obtain

$$\begin{aligned} 7x_3 - 8 &= -7 \\ 6x_3 + 10 &= 4. \end{aligned} \tag{8}$$

The first equation in (8) implies that $x_3 = 1/7$, and the second equation implies that $x_3 = -1$. This contradiction leads to the conclusion that there is no solution to the linear system (7). $\blacksquare$

Example 3.14. Show that there are infinitely many solutions to

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ 0x_2 + 7x_3 + 0x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6. \end{aligned} \tag{9}$$

Using the last equation in (9), we must have $x_4 = 2$, which is substituted into the second and third equations to get $x_3 = -1$, which checks out in both equations. But only two values x_3 and x_4 have been obtained from the second through fourth equations, and when they are substituted into the first equation of (9), the result is

$$x_2 = 4x_1 - 16, \tag{10}$$

which has infinitely many solutions; hence (9) has infinitely many solutions. If we choose a value of x_1 in (10), then the value of x_2 is uniquely determined. For example, if we include the equation $x_1 = 2$ in the system (9), then from (10) we compute $x_2 = -8$. ■

Theorem 3.4 states that the linear system $\boldsymbol{AX} = \boldsymbol{B}$, where $\boldsymbol{A}$ is an $N \times N$ matrix, has a unique solution if and only if $\det(\boldsymbol{A}) \neq 0$. The following theorem states that if any entry on the main diagonal of an upper- or lower-triangular matrix is zero, then $\det(\boldsymbol{A}) = 0$. Thus, by inspecting the coefficient matrices in the previous three examples, it is clear that the system in Example 3.12 has a unique solution, and the systems in Examples 3.13 and 3.14 do not have unique solutions. The proof of Theorem 3.6 can be found in most introductory linear algebra textbooks.

Theorem 3.6. If the $N \times N$ matrix $\boldsymbol{A} = [a_{ij}]$ is either upper or lower triangular, then

$$\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{NN} = \prod_{i=1}^{N} a_{ii}. \tag{11}$$

The value of the determinant for the coefficient matrix in Example 3.12 is $\det \boldsymbol{A} = 4(-2)(6)(3) = -144$. The values of the determinants of the coefficient matrices in Examples 3.13 and 3.14 are both $4(0)(6)(3) = 0$.

Numerical Methods Using Matlab, 4th Edition, 2004
John H. Mathews and Kurtis K. Fink
ISBN: 0-13-065248-2

Prentice-Hall Inc.
Upper Saddle River, New Jersey, USA
http://vig.prenhall.com/