Big "O" Truncation Error

$O(h^n)$ Order of Approximation

Clearly, the sequences $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ and $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ are both converging to zero. In addition, it should be observed that the first sequence is converging to zero more rapidly than the second sequence. In the coming chapters some special terminology and notation will be used to describe how rapidly a sequence is converging.

Definition 1.9. The function $f(h)$ is said to be ***big Oh*** of $g(h)$, denoted $f(h) = \boldsymbol{O}(g(h))$, if there exist constants C and c such that

$$|f(h)| \le C|g(h)| \quad \text{whenever } h \le c. \tag{7}$$ ▲

Example 1.20. Consider the functions $f(x) = x^2 + 1$ and $g(x) = x^3$. Since $x^2 \le x^3$ and $1 \le x^3$ for $x \ge 1$, it follows that $x^2 + 1 \le 2x^3$ for $x \ge 1$. Therefore, $f(x) = \boldsymbol{O}(g(x))$. ■

The big Oh notation provides a useful way of describing the rate of growth of a function in terms of well-known elementary functions (x^n, $x^{1/n}$, a^x, $\log_a x$, etc.).

The rate of convergence of sequences can be described in a similar manner.

Definition 1.10. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences. The sequence $\{x_n\}$ is said to be of order big Oh of $\{y_n\}$, denoted $x_n = \boldsymbol{O}(y_n)$, if there exist constants C and N such that

$$|x_n| \le C|y_n| \quad \text{whenever } n \ge N. \tag{8}$$ ▲

Example 1.21. $\dfrac{n^2-1}{n^3} = \boldsymbol{O}\left(\dfrac{1}{n}\right)$, since $\dfrac{n^2-1}{n^3} \le \dfrac{n^2}{n^3} = \dfrac{1}{n}$ whenever $n \ge 1$. ■

Often a function $f(h)$ is approximated by a function $p(h)$ and the error bound is known to be $M|h^n|$. This leads to the following definition.

Definition 1.11. Assume that $f(h)$ is approximated by the function $p(h)$ and that there exist a real constant $M > 0$ and a positive integer n so that

$$\frac{|f(h) - p(h)|}{|h^n|} \le M \quad \text{for sufficiently small } h. \tag{9}$$

We say that $p(h)$ ***approximates*** $f(h)$ with order of approximation $\boldsymbol{O}(h^n)$ and write

$$f(h) = p(h) + \boldsymbol{O}(h^n). \tag{10}$$ ▲

When relation (9) is rewritten in the form $|f(h) - p(h)| \le M|h^n|$, we see that the notation $\boldsymbol{O}(h^n)$ stands in place of the error bound $M|h^n|$. The following results show how to apply the definition to simple combinations of two functions.

Theorem 1.15. Assume that $f(h) = p(h) + \boldsymbol{O}(h^n)$, $g(h) = q(h) + \boldsymbol{O}(h^m)$, and $r = \min\{m, n\}$. Then

$$f(h) + g(h) = p(h) + q(h) + \boldsymbol{O}(h^r), \tag{11}$$

$$f(h)g(h) = p(h)q(h) + \boldsymbol{O}(h^r), \tag{12}$$

and

$$\frac{f(h)}{g(h)} = \frac{p(h)}{q(h)} + \boldsymbol{O}(h^r) \quad \text{provided that } g(h) \ne 0 \text{ and } q(h) \ne 0. \tag{13}$$

It is instructive to consider $p(x)$ to be the nth Taylor polynomial approximation of $f(x)$; then the remainder term is simply designated $\boldsymbol{O}(h^{n+1})$, which stands for the presence of omitted terms starting with the power h^{n+1}. The remainder term converges to zero with the same rapidity that h^{n+1} converges to zero as h approaches zero, as expressed in the relationship

$$\boldsymbol{O}(h^{n+1}) \approx Mh^{n+1} \approx \frac{f^{(n+1)}(c)}{(n+1)!}h^{n+1} \tag{14}$$

for sufficiently small h. Hence the notation $\boldsymbol{O}(h^{n+1})$ stands in place of the quantity Mh^{n+1}, where M is a constant or "behaves like a constant."

Theorem 1.16 (Taylor's Theorem). Assume that $f \in C^{n+1}[a, b]$. If both x_0 and $x = x_0 + h$ lie in $[a, b]$, then

$$f(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}h^k + \boldsymbol{O}(h^{n+1}). \tag{15}$$

The following example illustrates the theorems above. The computations use the addition properties (i) $\boldsymbol{O}(h^p) + \boldsymbol{O}(h^p) = \boldsymbol{O}(h^p)$, (ii) $\boldsymbol{O}(h^p) + \boldsymbol{O}(h^q) = \boldsymbol{O}(h^r)$, where $r = \min\{p, q\}$, and the multiplicative property (iii) $\boldsymbol{O}(h^p)\boldsymbol{O}(h^q) = \boldsymbol{O}(h^s)$, where $s = p + q$.

Example 1.22. Consider the Taylor polynomial expansions

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \boldsymbol{O}(h^4) \quad \text{and} \quad \cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \boldsymbol{O}(h^6).$$

Determine the order of approximation for their sum and product.

For the sum we have

$$\begin{aligned} e^h + \cos(h) &= 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \boldsymbol{O}(h^4) + 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \boldsymbol{O}(h^6) \\ &= 2 + h + \frac{h^3}{3!} + \boldsymbol{O}(h^4) + \frac{h^4}{4!} + \boldsymbol{O}(h^6). \end{aligned}$$

Since $\boldsymbol{O}(h^4) + \frac{h^4}{4!} = \boldsymbol{O}(h^4)$ and $\boldsymbol{O}(h^4) + \boldsymbol{O}(h^6) = \boldsymbol{O}(h^4)$, this reduces to

$$e^h + \cos(h) = 2 + h + \frac{h^3}{3!} + \boldsymbol{O}(h^4),$$

and the order of approximation is $\boldsymbol{O}(h^4)$.

The product is treated similarly:

$$\begin{aligned}
e^h\cos(h) &= \left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\boldsymbol{O}(h^4)\right)\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}+\boldsymbol{O}(h^6)\right)\\
&= \left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}\right)\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}\right)\\
&\quad+\left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}\right)\boldsymbol{O}(h^6)+\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}\right)\boldsymbol{O}(h^4)\\
&\quad+\boldsymbol{O}(h^4)\boldsymbol{O}(h^6)\\
&= 1+h-\frac{h^3}{3}-\frac{5h^4}{24}-\frac{h^5}{24}+\frac{h^6}{48}+\frac{h^7}{144}\\
&\quad+\boldsymbol{O}(h^6)+\boldsymbol{O}(h^4)+\boldsymbol{O}(h^4)\boldsymbol{O}(h^6).
\end{aligned}$$

Since $\boldsymbol{O}(h^4)\boldsymbol{O}(h^6)=\boldsymbol{O}(h^{10})$ and

$$-\frac{5h^4}{24}-\frac{h^5}{24}+\frac{h^6}{48}+\frac{h^7}{144}+\boldsymbol{O}(h^6)+\boldsymbol{O}(h^4)+\boldsymbol{O}(h^{10})=\boldsymbol{O}(h^4),$$

the preceding equation is simplified to yield

$$e^h\cos(h)=1+h-\frac{h^3}{3}+\boldsymbol{O}(h^4),$$

and the order of approximation is $\boldsymbol{O}(h^4)$. ■

Order of Convergence of a Sequence

Numerical approximations are often arrived at by computing a sequence of approximations that get closer and closer to the answer desired. The definition of big Oh for sequences was given in Definition 1.10, and the definition of order of convergence for a sequence is analogous to that given for functions in Definition 1.11.

Definition 1.12. Suppose that $\lim_{n\to\infty} x_n = x$ and $\{r_n\}_{n=1}^{\infty}$ is a sequence with $\lim_{n\to\infty} r_n = 0$. We say that $\{x_n\}_{n=1}^{\infty}$ ***converges*** to x with the order of convergence $\boldsymbol{O}(r_n)$, if there exists a constant $K>0$ such that

$$\frac{|x_n-x|}{|r_n|}\le K \qquad \text{for } n \text{ sufficiently large.}$$

This is indicated by writing $x_n = x+\boldsymbol{O}(r_n)$, or $x_n \to x$ with order of convergence $\boldsymbol{O}(r_n)$. ▲

Example 1.23. Let $x_n=\cos(n)/n^2$ and $r_n=1/n^2$; then $\lim_{n\to\infty} x_n = 0$ with a rate of convergence $\boldsymbol{O}(1/n^2)$. This follows immediately from the relation

$$\frac{|\cos(n)/n^2|}{|1/n^2|}=|\cos(n)|\le 1 \qquad \text{for all } n.$$

■

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