

Figure 2.6 The decision process for the bisection process.

Definition 2.3. Assume that $f(x)$ is a continuous function. Any number r for which $f(r) = 0$ is called a **root of the equation** $f(x) = 0$. Also, we say that r is a **zero of the function** $f(x)$. ▲

For example, the equation $2x^2 + 5x - 3 = 0$ has two real roots $r_1 = 0.5$ and $r_2 = -3$, whereas the corresponding function $f(x) = 2x^2 + 5x - 3 = (2x - 1)(x + 3)$ has two real zeros, $r_1 = 0.5$ and $r_2 = -3$.

Bisection Method of Bolzano

In this section we develop our first bracketing method for finding a zero of a continuous function. We must start with an initial interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs. Since the graph $y = f(x)$ of a continuous function is unbroken, it will

cross the x -axis at a zero $x = r$ that lies somewhere in the interval (see Figure 2.6). The bisection method systematically moves the endpoints of the interval closer and closer together until we obtain an interval of arbitrarily small width that brackets the zero. The decision step for this process of interval halving is first to choose the midpoint $c = (a + b)/2$ and then to analyze the three possibilities that might arise:

- (4) If $f(a)$ and $f(c)$ have opposite signs, a zero lies in $[a, c]$.
- (5) If $f(c)$ and $f(b)$ have opposite signs, a zero lies in $[c, b]$.
- (6) If $f(c) = 0$, then the zero is c .

If either case (4) or (5) occurs, we have found an interval half as wide as the original interval that contains the root, and we are “squeezing down on it” (see Figure 2.6). To continue the process, relabel the new smaller interval $[a, b]$ and repeat the process until the interval is as small as desired. Since the bisection process involves sequences of nested intervals and their midpoints, we will use the following notation to keep track of the details in the process:

$[a_0, b_0]$ is the starting interval and $c_0 = (a_0 + b_0)/2$ is the midpoint.

- (7) $[a_1, b_1]$ is the second interval, which brackets the zero r , and c_1 is its midpoint; the interval $[a_1, b_1]$ is half as wide as $[a_0, b_0]$.

After arriving at the n th interval $[a_n, b_n]$, which brackets r and has midpoint c_n , the interval $[a_{n+1}, b_{n+1}]$ is constructed, which also brackets r and is half as wide as $[a_n, b_n]$.

It is left as an exercise for the reader to show that the sequence of left endpoints is increasing and the sequence of right endpoints is decreasing; that is,

$$(8) \quad a_0 \leq a_1 \leq \cdots \leq a_n \leq \cdots \leq r \leq \cdots \leq b_n \leq \cdots \leq b_1 \leq b_0,$$

where $c_n = (a_n + b_n)/2$, and if $f(a_{n+1})f(b_{n+1}) < 0$, then

$$(9) \quad [a_{n+1}, b_{n+1}] = [a_n, c_n] \quad \text{or} \quad [a_{n+1}, b_{n+1}] = [c_n, b_n] \quad \text{for all } n.$$

Theorem 2.4 (Bisection Theorem). Assume that $f \in C[a, b]$ and that there exists a number $r \in [a, b]$ such that $f(r) = 0$. If $f(a)$ and $f(b)$ have opposite signs, and $\{c_n\}_{n=0}^{\infty}$ represents the sequence of midpoints generated by the bisection process of (8) and (9), then

$$(10) \quad |r - c_n| \leq \frac{b - a}{2^{n+1}} \quad \text{for } n = 0, 1, \dots,$$

and therefore the sequence $\{c_n\}_{n=0}^{\infty}$ converges to the zero $x = r$; that is,

$$(11) \quad \lim_{n \rightarrow \infty} c_n = r.$$

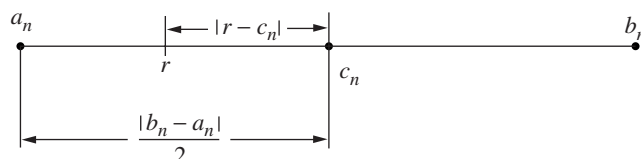


Figure 2.7 The root r and midpoint c_n of $[a_n, b_n]$ for the bisection method.

Proof. Since both the zero r and the midpoint c_n lie in the interval $[a_n, b_n]$, the distance between c_n and r cannot be greater than half the width of this interval (see Figure 2.7). Thus

$$(12) \quad |r - c_n| \leq \frac{b_n - a_n}{2} \quad \text{for all } n.$$

Observe that the successive interval widths form the pattern

$$\begin{aligned} b_1 - a_1 &= \frac{b_0 - a_0}{2^1}, \\ b_2 - a_2 &= \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}. \end{aligned}$$

It is left as an exercise for the reader to use mathematical induction and show that

$$(13) \quad b_n - a_n = \frac{b_0 - a_0}{2^n}.$$

Combining (12) and (13) results in

$$(14) \quad |r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}} \quad \text{for all } n.$$

Now an argument similar to the one given in Theorem 2.3 can be used to show that (14) implies that the sequence $\{c_n\}_{n=0}^{\infty}$ converges to r and the proof of the theorem is complete. •

Example 2.7. The function $h(x) = x \sin(x)$ occurs in the study of undamped forced oscillations. Find the value of x that lies in the interval $[0, 2]$, where the function takes on the value $h(x) = 1$ (the function $\sin(x)$ is evaluated in radians).

We use the bisection method to find a zero of the function $f(x) = x \sin(x) - 1$. Starting with $a_0 = 0$ and $b_0 = 2$, we compute

$$f(0) = -1.000000 \quad \text{and} \quad f(2) = 0.818595,$$

so a root of $f(x) = 0$ lies in the interval $[0, 2]$. At the midpoint $c_0 = 1$, we find that $f(1) = -0.158529$. Hence the function changes sign on $[c_0, b_0] = [1, 2]$.

Table 2.1 Bisection Method Solution of $x \sin(x) - 1 = 0$

k	Left endpoint, a_k	Midpoint, c_k	Right endpoint, b_k	Function value, $f(c_k)$
0	0	1.	2.	-0.158529
1	1.0	1.5	2.0	0.496242
2	1.00	1.25	1.50	0.186231
3	1.000	1.125	1.250	0.015051
4	1.0000	1.0625	1.1250	-0.071827
5	1.06250	1.09375	1.12500	-0.028362
6	1.093750	1.109375	1.125000	-0.006643
7	1.1093750	1.1171875	1.1250000	0.004208
8	1.10937500	1.11328125	1.11718750	-0.001216
\vdots	\vdots	\vdots	\vdots	\vdots

To continue, we squeeze from the left and set $a_1 = c_0$ and $b_1 = b_0$. The midpoint is $c_1 = 1.5$ and $f(c_1) = 0.496242$. Now, $f(1) = -0.158529$ and $f(1.5) = 0.496242$ imply that the root lies in the interval $[a_1, c_1] = [1.0, 1.5]$. The next decision is to squeeze from the right and set $a_2 = a_1$ and $b_2 = c_1$. In this manner we obtain a sequence $\{c_k\}$ that converges to $r \approx 1.114157141$. A sample calculation is given in Table 2.1. ■

A virtue of the bisection method is that formula (10) provides a predetermined estimate for the accuracy of the computed solution. In Example 2.7 the width of the starting interval was $b_0 - a_0 = 2$. Suppose that Table 2.1 were continued to the thirty-first iterate; then, by (10), the error bound would be $|E_{31}| \leq (2 - 0)/2^{32} \approx 4.656613 \times 10^{-10}$. Hence c_{31} would be an approximation to r with nine decimal places of accuracy. The number N of repeated bisections needed to guarantee that the N th midpoint c_N is an approximation to a zero and has an error less than the preassigned value δ is

$$(15) \quad N = \text{int} \left(\frac{\ln(b - a) - \ln(\delta)}{\ln(2)} \right).$$

The proof of this formula is left as an exercise.

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