

4.5 Chebyshev Polynomials

We now turn our attention to polynomial interpolation for $f(x)$ over $[-1, 1]$ based on the nodes $-1 \leq x_0 < x_1 < \cdots < x_N \leq 1$. Both the Lagrange and Newton polynomials satisfy

$$f(x) = P_N(x) + E_N(x),$$

where

$$(1) \quad E_N(x) = Q(x) \frac{f^{(N+1)}(c)}{(N+1)!}$$

and $Q(x)$ is the polynomial of degree $N+1$:

$$(2) \quad Q(x) = (x - x_0)(x - x_1) \cdots (x - x_N).$$

Using the relationship

$$|E_N(x)| \leq |Q(x)| \frac{\max_{-1 \leq x \leq 1} \{|f^{(N+1)}(x)|\}}{(N+1)!},$$

our task is to follow Chebyshev's derivation on how to select the set of nodes $\{x_k\}_{k=0}^N$ that minimizes $\max_{-1 \leq x \leq 1} \{|Q(x)|\}$. This leads us to a discussion of Chebyshev polynomials and some of their properties. To begin, the first eight Chebyshev polynomials are listed in Table 4.11.

Table 4.11 Chebyshev Polynomials
 $T_0(x)$ through $T_7(x)$

$T_0(x) = 1$
$T_1(x) = x$
$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$
$T_4(x) = 8x^4 - 8x^2 + 1$
$T_5(x) = 16x^5 - 20x^3 + 5x$
$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

Properties of Chebyshev Polynomials

Property 1. Recurrence Relation

Chebyshev polynomials can be generated in the following way. Set $T_0(x) = 1$ and $T_1(x) = x$ and use the recurrence relation

$$(3) \quad T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x) \quad \text{for } k = 2, 3, \dots$$

Property 2. Leading Coefficient

The coefficient of x^N in $T_N(x)$ is 2^{N-1} when $N \geq 1$.

Property 3. Symmetry

When $N = 2M$, $T_{2M}(x)$ is an even function, that is,

$$(4) \quad T_{2M}(-x) = T_{2M}(x).$$

When $N = 2M + 1$, $T_{2M+1}(x)$ is an odd function, that is,

$$(5) \quad T_{2M+1}(-x) = -T_{2M+1}(x).$$

Property 4. Trigonometric Representation on $[-1, 1]$

$$(6) \quad T_N(x) = \cos(N \arccos(x)) \quad \text{for } -1 \leq x \leq 1.$$

Property 5. Distinct Zeros in $[-1, 1]$

$T_N(x)$ has N distinct zeros x_k that lie in the interval $[-1, 1]$ (see Figure 4.15):

$$(7) \quad x_k = \cos\left(\frac{(2k+1)\pi}{2N}\right) \quad \text{for } k = 0, 1, \dots, N-1.$$

These values are called the *Chebyshev abscissas (nodes)*.

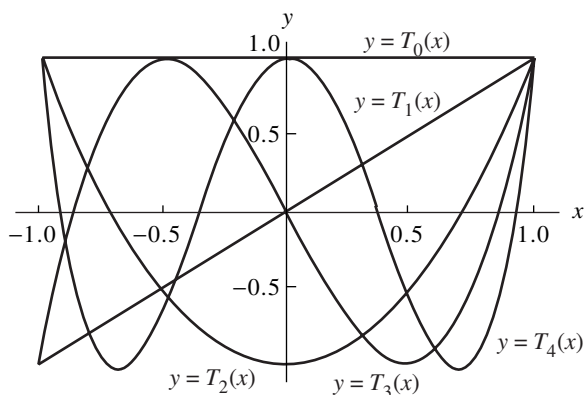


Figure 4.15 The graphs of the Chebyshev polynomials $T_0(x)$, $T_1(x)$, \dots , $T_4(x)$ over $[-1, 1]$.

Property 6. Extreme Values

$$(8) \quad |T_N(x)| \leq 1 \quad \text{for } -1 \leq x \leq 1.$$

Property 1 is often used as the definition for higher-order Chebyshev polynomials. Let us show that $T_3(x) = 2xT_2(x) - T_1(x)$. Using the expressions for $T_1(x)$ and $T_2(x)$ in Table 4.11, we obtain

$$2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x = T_3(x).$$

Property 2 is proved by observing that the recurrence relation doubles the leading coefficient of $T_{N-1}(x)$ to get the leading coefficient of $T_N(x)$.

Property 3 is established by showing that $T_{2M}(x)$ involves only even powers of x and $T_{2M+1}(x)$ involves only odd powers of x . The details are left for the reader.

The proof of property 4 uses the trigonometric identity

$$\cos(k\theta) = \cos(2\theta) \cos((k-2)\theta) - \sin(2\theta) \sin((k-2)\theta).$$

Substitute $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and get

$$\cos(k\theta) = 2\cos(\theta)(\cos(\theta)\cos((k-2)\theta) - \sin(\theta)\sin((k-2)\theta)) - \cos((k-2)\theta),$$

which is simplified as

$$\cos(k\theta) = 2\cos(\theta)\cos((k-1)\theta) - \cos((k-2)\theta).$$

Finally, substitute $\theta = \arccos(x)$ and obtain

$$(9) \quad \begin{aligned} 2x \cos((k-1)\arccos(x)) - \cos((k-2)\arccos(x)) \\ = \cos(k\arccos(x)) \quad \text{for } -1 \leq x \leq 1. \end{aligned}$$

The first two Chebyshev polynomials are $T_0(x) = \cos(0 \arccos(x)) = 1$ and $T_1(x) = \cos(1 \arccos(x)) = x$. Now assume that $T_k(x) = \cos(k \arccos(x))$ for $k = 2, 3, \dots, N - 1$. Formula (3) is used with (9) to establish the general case:

$$\begin{aligned} T_N(x) &= 2xT_{N-1}(x) - T_{N-2}(x) \\ &= 2x \cos((N-1) \arccos(x)) - \cos((N-2) \arccos(x)) \\ &= \cos(N \arccos(x)) \quad \text{for } -1 \leq x \leq 1. \end{aligned}$$

Properties 5 and 6 are consequences of property 4.

Minimax

The Russian mathematician Chebyshev studied how to minimize the upper bound for $|E_N(x)|$. One upper bound can be formed by taking the product of the maximum value of $|Q(x)|$ over all x in $[-1, 1]$ and the maximum value $|f^{(N+1)}(x)/(N+1)!$ over all x in $[-1, 1]$. To minimize the factor $\max\{|Q(x)|\}$, Chebyshev discovered that x_0, x_1, \dots, x_N should be chosen so that $Q(x) = (1/2^N)T_{N+1}(x)$.

Theorem 4.6. Assume that N is fixed. Among all possible choices for $Q(x)$ in equation (2), and thus among all possible choices for the distinct nodes $\{x_k\}_{k=0}^N$ in $[-1, 1]$, the polynomial $T(x) = T_{N+1}(x)/2^N$ is the unique choice that has the property

$$\max_{-1 \leq x \leq 1} \{|T(x)|\} \leq \max_{-1 \leq x \leq 1} \{|Q(x)|\}.$$

Moreover,

$$(10) \quad \max_{-1 \leq x \leq 1} \{|T(x)|\} = \frac{1}{2^N}.$$

The consequence of this result can be stated by saying that for Lagrange interpolation $f(x) = P_N(x) + E_N(x)$ on $[-1, 1]$, the minimum value of the error bound

$$(\max\{|Q(x)|\})(\max\{|f^{(N+1)}(x)/(N+1)!\})$$

is achieved when the nodes $\{x_k\}$ are the Chebyshev abscissas of $T_{N+1}(x)$. As an illustration, we look at the Lagrange coefficient polynomials that are used in forming $P_3(x)$. First we use equally spaced nodes and then the Chebyshev nodes. Recall that the Lagrange polynomial of degree $N = 3$ has the form

$$(11) \quad P_3(x) = f(x_0)L_{3,0}(x) + f(x_1)L_{3,1}(x) + f(x_2)L_{3,2}(x) + f(x_3)L_{3,3}(x).$$

Equally Spaced Nodes

If $f(x)$ is approximated by a polynomial of degree at most $N = 3$ on $[-1, 1]$, the equally spaced nodes $x_0 = -1, x_1 = -1/3, x_2 = 1/3,$ and $x_3 = 1$ are easy to use for calculations. Substitution of these values into formula (8) of Section 4.3 and simplifying will produce the coefficient polynomials $L_{3,k}(x)$ in Table 4.12.

Table 4.12 Lagrange Coefficient Polynomials Used to Form $P_3(x)$
Based on Equally Spaced Nodes $x_k = -1 + 2k/3$

$$\begin{aligned} L_{3,0}(x) &= -0.06250000 + 0.06250000x + 0.56250000x^2 - 0.56250000x^3 \\ L_{3,1}(x) &= 0.56250000 - 1.68750000x - 0.56250000x^2 + 1.68750000x^3 \\ L_{3,2}(x) &= 0.56250000 + 1.68750000x - 0.56250000x^2 - 1.68750000x^3 \\ L_{3,3}(x) &= -0.06250000 - 0.06250000x + 0.56250000x^2 + 0.56250000x^3 \end{aligned}$$

Table 4.13 Coefficient Polynomials Used to Form $P_3(x)$ Based on the
Chebyshev Nodes $x_k = \cos((7 - 2k)\pi/8)$

$$\begin{aligned} C_0(x) &= -0.10355339 + 0.11208538x + 0.70710678x^2 - 0.76536686x^3 \\ C_1(x) &= 0.60355339 - 1.57716102x - 0.70710678x^2 + 1.84775906x^3 \\ C_2(x) &= 0.60355339 + 1.57716102x - 0.70710678x^2 - 1.84775906x^3 \\ C_3(x) &= -0.10355339 - 0.11208538x + 0.70710678x^2 + 0.76536686x^3 \end{aligned}$$

Chebyshev Nodes

When $f(x)$ is to be approximated by a polynomial of degree at most $N = 3$, using the Chebyshev nodes $x_0 = \cos(7\pi/8)$, $x_1 = \cos(5\pi/8)$, $x_2 = \cos(3\pi/8)$, and $x_3 = \cos(\pi/8)$, the coefficient polynomials are tedious to find (but this can be done by a computer). The results after simplification are shown in Table 4.13.

Example 4.14. Compare the Lagrange polynomials of degree $N = 3$ for $f(x) = e^x$ that are obtained by using the coefficient polynomials in Tables 4.12 and 4.13, respectively.

Using equally spaced nodes, we get the polynomial

$$P(x) = 0.99519577 + 0.99904923x + 0.54788486x^2 + 0.17615196x^3.$$

This is obtained by finding the function values

$$\begin{aligned} f(x_0) &= e^{(-1)} = 0.36787944, & f(x_1) &= e^{(-1/3)} = 0.71653131, \\ f(x_2) &= e^{(1/3)} = 1.39561243, & f(x_3) &= e^{(1)} = 2.71828183, \end{aligned}$$

and using the coefficient polynomials $L_{3,k}(x)$ in Table 4.12, and forming the linear combination

$$\begin{aligned} P(x) &= 0.36787944L_{3,0}(x) + 0.71653131L_{3,1}(x) + 1.39561243L_{3,2}(x) \\ &\quad + 2.71828183L_{3,3}(x). \end{aligned}$$

Similarly, when the Chebyshev nodes are used, we obtain

$$V(x) = 0.99461532 + 0.99893323x + 0.54290072x^2 + 0.17517569x^3.$$

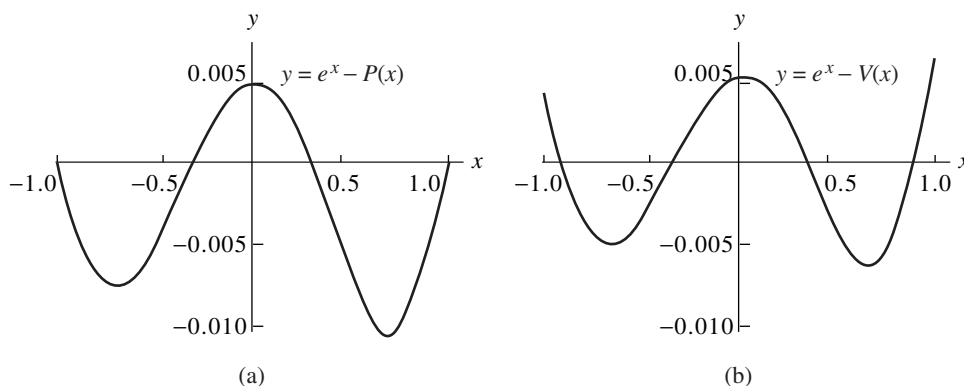


Figure 4.16 (a) The error function $y = e^x - P(x)$ for Lagrange approximation over $[-1, 1]$.
 (b) The error function $y = e^x - V(x)$ for Lagrange approximation over $[-1, 1]$.

Notice that the coefficients are different from those of $P(x)$. This is a consequence of using different nodes and function values:

$$\begin{aligned} f(x_0) &= e^{-0.92387953} = 0.39697597, \\ f(x_1) &= e^{-0.38268343} = 0.68202877, \\ f(x_2) &= e^{0.38268343} = 1.46621380, \\ f(x_3) &= e^{0.92387953} = 2.51904417. \end{aligned}$$

Then the alternative set of coefficient polynomials $C_k(x)$ in Table 4.13 is used to form the linear combination

$$V(x) = 0.39697597C_0(x) + 0.68202877C_1(x) + 1.46621380C_2(x) + 2.51904417C_3(x).$$

For a comparison of the accuracy of $P(x)$ and $V(x)$, the error functions are graphed in Figure 4.16(a) and (b), respectively. The maximum error $|e^x - P(x)|$ occurs at $x = 0.75490129$, and

$$|e^x - P(x)| \leq 0.00998481 \quad \text{for } -1 \leq x \leq 1.$$

The maximum error $|e^x - V(x)|$ occurs at $x = 1$, and we get

$$|e^x - V(x)| \leq 0.00665687 \quad \text{for } -1 \leq x \leq 1.$$

Notice that the maximum error in $V(x)$ is about two-thirds the maximum error in $P(x)$. Also, the error is spread out more evenly over the interval. ■

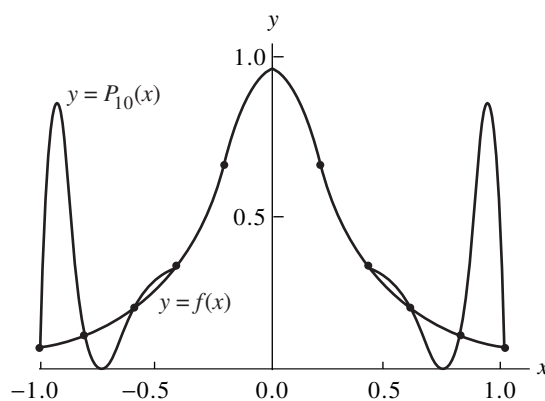


Figure 4.17 (a) The polynomial approximation to $y = 1/(1 + 12x^2)$ based on 11 equally spaced nodes over $[-1, 1]$.

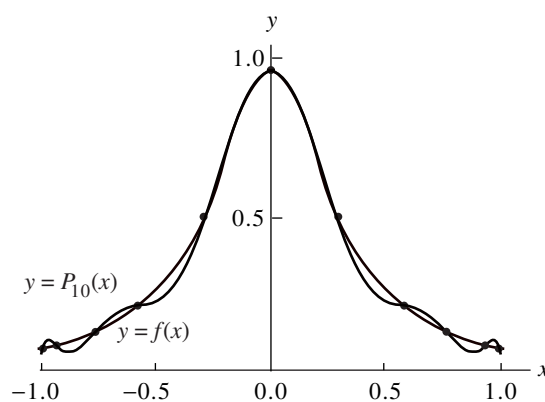


Figure 4.17 (b) The polynomial approximation to $y = 1/(1 + 12x^2)$ based on 11 Chebyshev nodes over $[-1, 1]$.

Runge Phenomenon

We now look deeper to see the advantage of using the Chebyshev interpolation nodes. Consider Lagrange interpolating to $f(x)$ over the interval $[-1, 1]$ based on equally spaced nodes. Does the error $E_N(x) = f(x) - P_N(x)$ tend to zero as N increases? For functions like $\sin(x)$ or e^x , where all the derivatives are bounded by the same constant M , the answer is yes. In general, the answer to this question is no, and it is easy to find functions for which the sequence $\{P_N(x)\}$ does not converge. If $f(x) = 1/(1 + 12x^2)$, the maximum of the error term $E_N(x)$ grows when $N \rightarrow \infty$. This nonconvergence is called the **Runge phenomenon**. The Lagrange polynomial of degree 10 based on 11 equally spaced nodes for this function is shown in Figure 4.17(a). Wild oscillations occur near the end of the interval. If the number of nodes is increased, then the oscillations become larger. This problem occurs because the nodes are equally spaced!

If the Chebyshev nodes are used to construct an interpolating polynomial of degree 10 to $f(x) = 1/(1 + 12x^2)$, the error is much smaller, as seen in Figure 4.17(b). Under the condition that Chebyshev nodes be used, the error $E_N(x)$ will go to zero

as $N \rightarrow \infty$. In general, if $f(x)$ and $f'(x)$ are continuous on $[-1, 1]$, then it can be proved that Chebyshev interpolation will produce a sequence of polynomials $\{P_N(x)\}$ that converges uniformly to $f(x)$ over $[-1, 1]$.

Transforming the Interval

Sometimes it is necessary to take a problem stated on an interval $[a, b]$ and reformulate the problem on the interval $[c, d]$ where the solution is known. If the approximation $P_N(x)$ to $f(x)$ is to be obtained on the interval $[a, b]$, then we change the variable so that the problem is reformulated on $[-1, 1]$:

$$(12) \quad x = \left(\frac{b-a}{2}\right)t + \frac{a+b}{2} \quad \text{or} \quad t = 2\frac{x-a}{b-a} - 1,$$

where $a \leq x \leq b$ and $-1 \leq t \leq 1$.

The required Chebyshev nodes of $T_{N+1}(t)$ on $[-1, 1]$ are

$$(13) \quad t_k = \cos\left((2N+1-2k)\frac{\pi}{2N+2}\right) \quad \text{for } k = 0, 1, \dots, N$$

and the interpolating nodes on $[a, b]$ are obtained by using (12):

$$(14) \quad x_k = t_k \frac{b-a}{2} + \frac{a+b}{2} \quad \text{for } k = 0, 1, \dots, N.$$

Theorem 4.7 (Lagrange-Chebyshev Approximation Polynomial). Assume that $P_N(x)$ is the Lagrange polynomial that is based on the Chebyshev nodes given in (14). If $f \in C^{N+1}[a, b]$, then

$$(15) \quad |f(x) - P_N(x)| \leq \frac{2(b-a)^{N+1}}{4^{N+1}(N+1)!} \max_{a \leq x \leq b} \{|f^{(N+1)}(x)|\}.$$

Example 4.15. For $f(x) = \sin(x)$ on $[0, \pi/4]$, find the Chebyshev nodes and the error bound (15) for the Lagrange polynomial $P_5(x)$.

Formulas (12), (13), and (14) are used to find the nodes;

$$x_k = \cos\left(\frac{(11-2k)\pi}{12}\right) \frac{\pi}{8} + \frac{\pi}{8} \quad \text{for } k = 0, 1, \dots, 5.$$

Using the bound $|f^{(6)}(x)| \leq |-\sin(\pi/4)| = 2^{-1/2} = M$ in (15), we get

$$|f(x) - P_N(x)| \leq \left(\frac{\pi}{8}\right)^6 \left(\frac{2}{6!}\right) 2^{-1/2} \leq 0.00000720. \quad \blacksquare$$

Orthogonal Property

In Example 4.14, the Chebyshev nodes were used to find the Lagrange interpolating polynomial. In general, this implies that the Chebyshev polynomial of degree N can be obtained by Lagrange interpolation based on the $N + 1$ nodes that are the $N + 1$ zeros of $T_{N+1}(x)$. However, a direct approach to finding the approximation polynomial is to express $P_N(x)$ as a linear combination of the polynomials $T_k(x)$, which were given in Table 4.11. Therefore, the Chebyshev interpolating polynomial can be written in the form

$$(16) \quad P_N(x) = \sum_{k=0}^N c_k T_k(x) = c_0 T_0(x) + c_1 T_1(x) + \cdots + c_N T_N(x).$$

The coefficients $\{c_k\}$ in (16) are easy to find. The technical proof requires the use of the following orthogonality properties. Let

$$(17) \quad x_k = \cos\left(\pi \frac{2k+1}{2N+2}\right) \quad \text{for } k = 0, 1, \dots, N;$$

$$(18) \quad \sum_{k=0}^N T_i(x_k) T_j(x_k) = 0 \quad \text{when } i \neq j,$$

$$(19) \quad \sum_{k=0}^N T_i(x_k) T_j(x_k) = \frac{N+1}{2} \quad \text{when } i = j \neq 0,$$

$$(20) \quad \sum_{k=0}^N T_0(x_k) T_0(x_k) = N + 1.$$

Property 4 and the identities (18) and (20) can be used to prove the following theorem.

Theorem 4.8 (Chebyshev Approximation). The Chebyshev approximation polynomial $P_N(x)$ of degree $\leq N$ for $f(x)$ over $[-1, 1]$ can be written as a sum of $\{T_j(x)\}$:

$$(21) \quad f(x) \approx P_N(x) = \sum_{j=0}^N c_j T_j(x).$$

The coefficients $\{c_j\}$ are computed with the formulas

$$(22) \quad c_0 = \frac{1}{N+1} \sum_{k=0}^N f(x_k) T_0(x_k) = \frac{1}{N+1} \sum_{k=0}^N f(x_k)$$

and

$$(23) \quad \begin{aligned} c_j &= \frac{2}{N+1} \sum_{k=0}^N f(x_k) T_j(x_k) \\ &= \frac{2}{N+1} \sum_{k=0}^N f(x_k) \cos\left(\frac{j\pi(2k+1)}{2N+2}\right) \quad \text{for } j = 1, 2, \dots, N. \end{aligned}$$

Example 4.16. Find the Chebyshev polynomial $P_3(x)$ that approximates the function $f(x) = e^x$ over $[-1, 1]$.

The coefficients are calculated using formulas (22) and (23), and the nodes $x_k = \cos(\pi(2k+1)/8)$ for $k = 0, 1, 2, 3$.

$$\begin{aligned} c_0 &= \frac{1}{4} \sum_{k=0}^3 e^{x_k} T_0(x_k) = \frac{1}{4} \sum_{k=0}^3 e^{x_k} = 1.26606568, \\ c_1 &= \frac{1}{2} \sum_{k=0}^3 e^{x_k} T_1(x_k) = \frac{1}{2} \sum_{k=0}^3 e^{x_k} x_k = 1.13031500, \\ c_2 &= \frac{1}{2} \sum_{k=0}^3 e^{x_k} T_2(x_k) = \frac{1}{2} \sum_{k=0}^3 e^{x_k} \cos\left(2\pi \frac{2k+1}{8}\right) = 0.27145036, \\ c_3 &= \frac{1}{2} \sum_{k=0}^3 e^{x_k} T_3(x_k) = \frac{1}{2} \sum_{k=0}^3 e^{x_k} \cos\left(3\pi \frac{2k+1}{8}\right) = 0.04379392. \end{aligned}$$

Therefore, the Chebyshev polynomial $P_3(x)$ for e^x is

$$(24) \quad \begin{aligned} P_3(x) &= 1.26606568T_0(x) + 1.13031500T_1(x) \\ &\quad + 0.27145036T_2(x) + 0.04379392T_3(x). \end{aligned}$$

If the Chebyshev polynomial (24) is expanded in powers of x , the result is

$$P_3(x) = 0.99461532 + 0.99893324x + 0.54290072x^2 + 0.17517568x^3,$$

which is the same as the polynomial $V(x)$ in Example 4.14. If the goal is to find the Chebyshev polynomial, formulas (22) and (23) are preferred. ■

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