9.9 Finite-Difference Method

Methods involving difference quotient approximations for derivatives can be used for solving certain second-order boundary value problems. Consider the linear equation

$$x'' = p(t)x'(t) + q(t)x(t) + r(t) \tag{1}$$

over $[a, b]$ with $x(a) = \alpha$ and $x(b) = \beta$. Form a partition of $[a, b]$ using the points $a = t_0 < t_1 < \cdots < t_N = b$, where $h = (b - a)/N$ and $t_j = a + jh$ for $j = 0, 1, \ldots, N$. The central-difference formulas discussed in Chapter 6 are used to approximate the derivatives

$$x'(t_j) = \frac{x(t_{j+1}) - x(t_{j-1})}{2h} + \boldsymbol{O}(h^2) \tag{2}$$

and

$$x''(t_j) = \frac{x(t_{j+1}) - 2x(t_j) + x(t_{j-1})}{h^2} + \boldsymbol{O}(h^2). \tag{3}$$

To start the derivation, we replace each term $x(t_j)$ on the right side of (2) and (3) with x_j, and the resulting equations are substituted into (1) to obtain the relation

$$\begin{aligned} \frac{x_{j+1} - 2x_j + x_{j-1}}{h^2} + \boldsymbol{O}(h^2) = p(t_j)\left(\frac{x_{j+1} - x_{j-1}}{2h} + \boldsymbol{O}(h^2)\right) \\ + q(t_j)x_j + r(t_j). \end{aligned} \tag{4}$$

Next, we drop the two terms $\boldsymbol{O}(h^2)$ in (4) and introduce the notation $p_j = p(t_j)$, $q_j = q(t_j)$, and $r_j = r(t_j)$; this produces the difference equation

$$\frac{x_{j+1} - 2x_j + x_{j-1}}{h^2} = p_j \frac{x_{j+1} - x_{j-1}}{2h} + q_j x_j + r_j, \tag{5}$$

which is used to compute numerical approximations to the differential equation (1). This is carried out by multiplying each side of (5) by h^2 and then collecting terms involving x_{j-1}, x_j, and x_{j+1} and arranging them in a system of linear equations:

$$\left(\frac{-h}{2}p_j - 1\right)x_{j-1} + (2 + h^2 q_j)x_j + \left(\frac{h}{2}p_j - 1\right)x_{j+1} = -h^2 r_j, \tag{6}$$

for $j = 1, 2, \ldots, N-1$, where $x_0 = \alpha$ and $x_N = \beta$. The system in (6) has the familiar tridiagonal form, which is more visible when displayed with matrix notation:

$$\begin{bmatrix} 2+h^2q_1 & \frac{h}{2}p_1 - 1 & & & & \\ \frac{-h}{2}p_2 - 1 & 2+h^2q_2 & \frac{h}{2}p_2 - 1 & & & \boldsymbol{O} \\ & \frac{-h}{2}p_j - 1 & 2+h^2q_j & \frac{h}{2}p_j - 1 & & \\ \boldsymbol{O} & & \frac{-h}{2}p_{N-2} - 1 & 2+h^2q_{N-2} & \frac{h}{2}p_{N-2} - 1 \\ & & & \frac{-h}{2}p_{N-1} - 1 & 2+h^2q_{N-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_j \\ x_{N-2} \\ x_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} -h^2r_1 + e_0 \\ -h^2r_2 \\ -h^2r_j \\ -h^2r_{N-2} \\ -h^2r_{N-1} + e_N \end{bmatrix},$$

where

$$e_0 = \left(\frac{h}{2}p_1 + 1\right)\alpha \quad \text{and} \quad e_N = \left(\frac{-h}{2}p_{N-1} + 1\right)\beta.$$

When computations with step size h are used, the numerical approximation to the solution is a set of discrete points $\{(t_j, x_j)\}$; if the analytic solution $x(t_j)$ is known, we can compare x_j and $x(t_j)$.

Example 9.18. Solve the boundary value problem

$$x''(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2}x(t) + 1$$

with $x(0) = 1.25$ and $x(4) = -0.95$ over the interval $[0, 4]$.

The functions p, q, and r are $p(t) = 2t/(1+t^2)$, $q(t) = -2/(1+t^2)$, and $r(t) = 1$, respectively. The finite-difference method is used to construct numerical solutions $\{x_j\}$ using the system of equations (6). Sample values of the approximations $\{x_{j,1}\}$, $\{x_{j,2}\}$, $\{x_{j,3}\}$, and $\{x_{j,4}\}$ corresponding to the step sizes $h_1 = 0.2$, $h_2 = 0.1$, $h_3 = 0.05$, and $h_4 = 0.025$ are given in Table 9.17. Figure 9.26 shows the graph of the polygonal path formed from $\{(t_j, x_{j,1})\}$ for the case $h_1 = 0.2$. There are 41 terms in the sequence generated with $h_2 = 0.1$, and the sequence $\{x_{j,2}\}$ only includes every other term from these computations; they correspond to the 21 values of $\{t_j\}$ given in Table 9.17. Similarly, the sequences $\{x_{j,3}\}$ and $\{x_{j,4}\}$ are a portion of the values generated with step sizes $h_3 = 0.05$ and $h_4 = 0.025$, respectively, and they correspond to the 21 values of $\{t_j\}$ in Table 9.17.

Next we compare numerical solutions in Table 9.17 with the analytic solution: $x(t) = 1.25 + 0.486089652t - 2.25t^2 + 2t\arctan(t) - \frac{1}{2}\ln(1+t^2) + \frac{1}{2}t^2\ln(1+t^2)$. The numerical

Table 9.17 Numerical Approximations for $x''(t) = \dfrac{2t}{1+t^2}x'(t) - \dfrac{2}{1+t^2}x(t) + 1$

t_j	$x_{j,1}$ $h = 0.2$	$x_{j,2}$ $h = 0.1$	$x_{j,3}$ $h = 0.05$	$x_{j,4}$ $h = 0.025$	$x(t_j)$ exact
0.0	1.250000	1.250000	1.250000	1.250000	1.250000
0.2	1.314503	1.316646	1.317174	1.317306	1.317350
0.4	1.320607	1.325045	1.326141	1.326414	1.326505
0.6	1.272755	1.279533	1.281206	1.281623	1.281762
0.8	1.177399	1.186438	1.188670	1.189227	1.189412
1.0	1.042106	1.053226	1.055973	1.056658	1.056886
1.2	0.874878	0.887823	0.891023	0.891821	0.892086
1.4	0.683712	0.698181	0.701758	0.702650	0.702947
1.6	0.476372	0.492027	0.495900	0.496865	0.497187
1.8	0.260264	0.276749	0.280828	0.281846	0.282184
2.0	0.042399	0.059343	0.063537	0.064583	0.064931
2.2	−0.170616	−0.153592	−0.149378	−0.148327	−0.147977
2.4	−0.372557	−0.355841	−0.351702	−0.350669	−0.350325
2.6	−0.557565	−0.541546	−0.537580	−0.536590	−0.536261
2.8	−0.720114	−0.705188	−0.701492	−0.700570	−0.700262
3.0	−0.854988	−0.841551	−0.838223	−0.837393	−0.837116
3.2	−0.957250	−0.945700	−0.942839	−0.942125	−0.941888
3.4	−1.022221	−1.012958	−1.010662	−1.010090	−1.009899
3.6	−1.045457	−1.038880	−1.037250	−1.036844	−1.036709
3.8	−1.022727	−1.019238	−1.018373	−1.018158	−1.018086
4.0	−0.950000	−0.950000	−0.950000	−0.950000	−0.950000

solutions can be shown to have error of order $\boldsymbol{O}(h^2)$. Hence reducing the step size by a factor of $\frac{1}{2}$ results in the error being reduced by about $\frac{1}{4}$. A careful scrutiny of Table 9.18 will reveal that this is happening. For instance, at $t_j = 1.0$ the errors incurred with step sizes h_1, h_2, h_3, and h_4 are $e_{j,1} = 0.014780$, $e_{j,2} = 0.003660$, $e_{j,3} = 0.000913$, and $e_{j,4} = 0.000228$, respectively. Their successive ratios $e_{j,2}/e_{j,1} = 0.003660/0.014780 = 0.2476$, $e_{j,3}/e_{j,2} = 0.000913/0.003660 = 0.2495$, and $e_{j,4}/e_{j,3} = 0.000228/0.000913 = 0.2497$ are approaching $\frac{1}{4}$.

Finally, we show how Richardson's improvement scheme can be used to extrapolate the seemingly inaccurate sequences $\{x_{j,1}\}$, $\{x_{j,2}\}$, $\{x_{j,3}\}$, and $\{x_{j,4}\}$ and obtain six digits of precision. Eliminate the error terms $\boldsymbol{O}(h^2)$ and $\boldsymbol{O}((h/2)^2)$ in the approximations $\{x_{j,1}\}$ and $\{x_{j,2}\}$ by generating the extrapolated sequence $\{z_{j,1}\} = \{(4x_{j,2} - x_{j,1})/3\}$. Similarly, the error terms $\boldsymbol{O}((h/2)^2)$ and $\boldsymbol{O}((h/4)^2)$ for $\{x_{j,2}\}$ and $\{x_{j,3}\}$ are eliminated by generating $\{z_{j,2}\} = \{(4x_{j,3} - x_{j,2})/3\}$. It has been shown that the second level of Richardson's improvement scheme applies to the sequences $\{z_{j,1}\}$ and $\{z_{j,2}\}$, so the third improvement is $\{(16z_{j,2} - z_{j,1})/15\}$. Let us illustrate the situation by finding the extrapolated values that

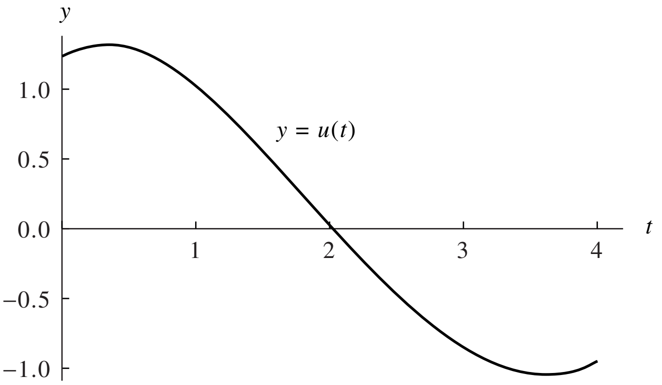

Figure 9.26 The graph of the numerical approximation for $x(t) = u(t) + w(t)$, which is the solution to

$$x''(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2}x(t) + 1$$

(using $h = 0.2$).

correspond to $t_j = 1.0$. The first extrapolated value is

$$\frac{4x_{j,2} - x_{j,1}}{3} = \frac{4(1.053226) - 1.042106}{3} = 1.056932 = z_{j,1}.$$

The second extrapolated value is

$$\frac{4x_{j,3} - x_{j,2}}{3} = \frac{4(1.055973) - 1.053226}{3} = 1.056889 = z_{j,2}.$$

Finally, the third extrapolation involves the terms $z_{j,1}$ and $z_{j,2}$:

$$\frac{16z_{j,2} - z_{j,1}}{15} = \frac{16(1.056889) - 1.056932}{15} = 1.056886.$$

This last computation contains six decimal places of accuracy. The values at the other points are given in Table 9.19. ■

Program 9.12 will call Program 9.11 to solve the tridiagonal system (6). Program 9.12 requires that the coefficient functions $p(t)$, $q(t)$, and $r(t)$ (boundary value problem (1)) be saved in M-files `p.m`, `q.m`, and `r.m`, respectively.

Program 9.11 (Tridiagonal Systems). To solve the tridiagonal system $\boldsymbol{CX} = \boldsymbol{B}$, where $\boldsymbol{C}$ is a tridiagonal matrix.

```
function X=trisys(A,D,C,B)
```

Table 9.18 Errors in Numerical Approximations Using the Finite-Difference Method

t_j	$x(t_j) - x_{j,1}$ $= e_{j,1}$	$x(t_j) - x_{j,2}$ $= e_{j,2}$	$x(t_j) - x_{j,3}$ $= e_{j,3}$	$x(t_j) - x_{j,4}$ $= e_{j,4}$
	$h_1 = 0.2$	$h_2 = 0.1$	$h_3 = 0.05$	$h_4 = 0.025$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.002847	0.000704	0.000176	0.000044
0.4	0.005898	0.001460	0.000364	0.000091
0.6	0.009007	0.002229	0.000556	0.000139
0.8	0.012013	0.002974	0.000742	0.000185
1.0	0.014780	0.003660	0.000913	0.000228
1.2	0.017208	0.004263	0.001063	0.000265
1.4	0.019235	0.004766	0.001189	0.000297
1.6	0.020815	0.005160	0.001287	0.000322
1.8	0.021920	0.005435	0.001356	0.000338
2.0	0.022533	0.005588	0.001394	0.000348
2.2	0.022639	0.005615	0.001401	0.000350
2.4	0.022232	0.005516	0.001377	0.000344
2.6	0.021304	0.005285	0.001319	0.000329
2.8	0.019852	0.004926	0.001230	0.000308
3.0	0.017872	0.004435	0.001107	0.000277
3.2	0.015362	0.003812	0.000951	0.000237
3.4	0.012322	0.003059	0.000763	0.000191
3.6	0.008749	0.002171	0.000541	0.000135
3.8	0.004641	0.001152	0.000287	0.000072
4.0	0.000000	0.000000	0.000000	0.000000

Table 9.19 Extrapolation of the Numerical Approximations $\{x_{j,1}\}$, $\{x_{j,2}\}$, $\{x_{j,3}\}$ Obtained with the Finite-Difference Method

t_j	$\frac{4x_{j,2}-x_{j,1}}{3}$ $= z_{j,1}$	$\frac{4x_{j,3}-x_{j,2}}{3}$ $= z_{j,2}$	$\frac{16z_{j,2}-z_{j,1}}{3}$	$x(t_j)$ Exact solution
0.0	1.250000	1.250000	1.250000	1.250000
0.2	1.317360	1.317351	1.317350	1.317350
0.4	1.326524	1.326506	1.326504	1.326505
0.6	1.281792	1.281764	1.281762	1.281762
0.8	1.189451	1.189414	1.189412	1.189412
1.0	1.056932	1.056889	1.056886	1.056886
1.2	0.892138	0.892090	0.892086	0.892086
1.4	0.703003	0.702951	0.702947	0.702948
1.6	0.497246	0.497191	0.497187	0.497187
1.8	0.282244	0.282188	0.282184	0.282184
2.0	0.064991	0.064935	0.064931	0.064931
2.2	−0.147918	−0.147973	−0.147977	−0.147977
2.4	−0.350268	−0.350322	−0.350325	−0.350325
2.6	−0.536207	−0.536258	−0.536261	−0.536261
2.8	−0.700213	−0.700259	−0.700263	−0.700262
3.0	−0.837072	−0.837113	−0.837116	−0.837116
3.2	−0.941850	−0.941885	−0.941888	−0.941888
3.4	−1.009870	−1.009898	−1.009899	−1.009899
3.6	−1.036688	−1.036707	−1.036708	−1.036708
3.8	−1.018075	−1.018085	−1.018086	−1.018086
4.0	−0.950000	−0.950000	−0.950000	−0.950000

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