Jacobi and Gauss-Seidel Iteration

3.6 Iterative Methods for Linear Systems

The goal of this section is to extend some of the iterative methods introduced in Chapter 2 to higher dimensions. We consider an extension of fixed-point iteration that applies to systems of linear equations.

Jacobi Iteration

Example 3.26. Consider the system of equations

$$\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y + 5z &= 15. \end{aligned} \tag{1}$$

These equations can be written in the form

$$\begin{aligned} x &= \frac{7 + y - z}{4} \\ y &= \frac{21 + 4x + z}{8} \\ z &= \frac{15 + 2x - y}{5}. \end{aligned} \tag{2}$$

Table 3.2 Convergent Jacobi Iteration for the Linear System (1)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.375	3.0
2	1.84375	3.875	3.025
3	1.9625	3.925	2.9625
4	1.99062500	3.97656250	3.00000000
5	1.99414063	3.99531250	3.00093750
⋮	⋮	⋮	⋮
15	1.99999993	3.99999985	2.99999993
⋮	⋮	⋮	⋮
19	2.00000000	4.00000000	3.00000000

This suggests the following Jacobi iterative process:

$$\begin{aligned} x_{k+1} &= \frac{7 + y_k - z_k}{4} \\ y_{k+1} &= \frac{21 + 4x_k + z_k}{8} \\ z_{k+1} &= \frac{15 + 2x_k - y_k}{5}. \end{aligned} \tag{3}$$

Let us show that if we start with $\boldsymbol{P}_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then the iteration in (3) appears to converge to the solution (2, 4, 3).

Substitute $x_0 = 1$, $y_0 = 2$, and $z_0 = 2$ into the right-hand side of each equation in (3) to obtain the new values

$$\begin{aligned} x_1 &= \frac{7 + 2 - 2}{4} = 1.75 \\ y_1 &= \frac{21 + 4 + 2}{8} = 3.375 \\ z_1 &= \frac{15 + 2 - 2}{5} = 3.00. \end{aligned}$$

The new point $\boldsymbol{P}_1 = (1.75, 3.375, 3.00)$ is closer to (2, 4, 3) than $\boldsymbol{P}_0$. Iteration using (3) generates a sequence of points $\{\boldsymbol{P}_k\}$ that converges to the solution (2, 4, 3) (see Table 3.2). ■

This process is called ***Jacobi iteration*** and can be used to solve certain types of linear systems. After 19 steps, the iteration has converged to the nine-digit machine approximation (2.00000000, 4.00000000, 3.00000000).

Linear systems with as many as 100,000 variables often arise in the solution of partial differential equations. The coefficient matrices for these systems are sparse;

that is, a large percentage of the entries of the coefficient matrix are zero. If there is a pattern to the nonzero entries (i.e., tridiagonal systems), then an iterative process provides an efficient method for solving these large systems.

Sometimes the Jacobi method does not work. Let us experiment and see that a rearrangement of the original linear system can result in a system of iteration equations that will produce a divergent sequence of points.

Example 3.27. Let the linear system (1) be rearranged as follows:

$$\begin{aligned} -2x + \ \ y + 5z &= \ \ 15 \\ 4x - 8y + \ \ z &= -21 \\ 4x - \ \ y + \ \ z &= \ \ \ 7. \end{aligned} \tag{4}$$

These equations can be written in the form

$$\begin{aligned} x &= \frac{-15 + y + 5z}{3} \\ y &= \frac{21 + 4x + z}{8} \\ z &= 7 - 4x + y. \end{aligned} \tag{5}$$

This suggests the following Jacobi iterative process:

$$\begin{aligned} x_{k+1} &= \frac{-15 + y_k + 5z_k}{3} \\ y_{k+1} &= \frac{21 + 4x_k + z_k}{8} \\ z_{k+1} &= 7 - 4x_k + y_k. \end{aligned} \tag{6}$$

See that if we start with $\boldsymbol{P}_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then the iteration using (6) will diverge away from the solution $(2, 4, 3)$.

Substitute $x_0 = 1$, $y_0 = 2$, and $z_0 = 2$ into the right-hand side of each equation in (6) to obtain the new values x_1, y_1, and z_1:

$$\begin{aligned} x_1 &= \frac{-15 + 2 + 10}{2} = -1.5 \\ y_1 &= \frac{21 + 4 + 2}{8} = 3.375 \\ z_1 &= 7 - 4 + 2 = 5.00. \end{aligned}$$

The new point $\boldsymbol{P}_1 = (-1.5, 3.375, 5.00)$ is farther away from the solution $(2, 4, 3)$ than $\boldsymbol{P}_0$. Iteration using the equations in (6) produces a divergent sequence (see Table 3.3). ■

Table 3.3 Divergent Jacobi Iteration for the Linear System (4)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	−1.5	3.375	5.0
2	6.6875	2.5	16.375
3	34.6875	8.015625	−17.25
4	−46.617188	17.8125	−123.73438
5	−307.929688	−36.150391	211.28125
6	502.62793	−124.929688	1202.56836
⋮	⋮	⋮	⋮

Gauss-Seidel Iteration

Sometimes the convergence can be speeded up. Observe that the Jacobi iterative process (3) yields three sequences $\{x_k\}$, $\{y_k\}$, and $\{z_k\}$ that converge to 2, 4, and 3, respectively (see Table 3.2). Since x_{k+1} is expected to be a better approximation to x than x_k, it seems reasonable that x_{k+1} could be used in place of x_k in the computation of y_{k+1}. Similarly, x_{k+1} and y_{k+1} might be used in the computation of z_{k+1}. The next example shows what happens when this is applied to the equations in Example 3.26.

Example 3.28. Consider the system of equations given in (1) and the Gauss-Seidel iterative process suggested by (2):

$$
\begin{aligned}
x_{k+1} &= \frac{7 + y_k - z_k}{4} \\
y_{k+1} &= \frac{21 + 4x_{k+1} + z_k}{8} \\
z_{k+1} &= \frac{15 + 2x_{k+1} - y_{k+1}}{5}.
\end{aligned} \tag{7}
$$

See that if we start with $\boldsymbol{P}_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then iteration using (7) will converge to the solution $(2, 4, 3)$.

Substitute $y_0 = 2$ and $z_0 = 2$ into the first equation of (7) and obtain

$$x_1 = \frac{7 + 2 - 2}{4} = 1.75.$$

Then substitute $x_1 = 1.75$ and $z_0 = 2$ into the second equation and get

$$y_1 = \frac{21 + 4(1.75) + 2}{8} = 3.75.$$

Finally, substitute $x_1 = 1.75$ and $y_1 = 3.75$ into the third equation to get

$$z_1 = \frac{15 + 2(1.75) - 3.75}{5} = 2.95.$$

Table 3.4 Convergent Gauss-Seidel Iteration for the System (1)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.75	2.95
2	1.95	3.96875	2.98625
3	1.995625	3.99609375	2.99903125
⋮	⋮	⋮	⋮
8	1.99999983	3.99999988	2.99999996
9	1.99999998	3.99999999	3.00000000
10	2.00000000	4.00000000	3.00000000

The new point $\boldsymbol{P}_1 = (1.75, 3.75, 2.95)$ is closer to (2, 4, 3) than $\boldsymbol{P}_0$ and is better than the value given in Example 3.26. Iteration using (7) generates a sequence $\{\boldsymbol{P}_k\}$ that converges to (2, 4, 3) (see Table 3.4). ■

In view of Examples 3.26 and 3.27, it is necessary to have some criterion to determine whether the Jacobi iteration will converge. Hence we make the following definition.

Definition 3.6. A matrix $\boldsymbol{A}$ of dimension $N \times N$ is said to be ***strictly diagonally dominant*** provided that

(8) $$|a_{kk}| > \sum_{\substack{j=1 \\ j \neq k}}^{N} |a_{kj}| \quad \text{for } k = 1, 2, \ldots, N.$$ ▲

This means that in each row of the matrix the magnitude of the element on the main diagonal must exceed the sum of the magnitudes of all other elements in the row. The coefficient matrix of the linear system (1) in Example 3.26 is strictly diagonally dominant because

$$\begin{aligned} &\text{In row 1:} \quad |4| > |-1| + |1| \\ &\text{In row 2:} \quad |-8| > |4| + |1| \\ &\text{In row 3:} \quad |5| > |-2| + |1|. \end{aligned}$$

All the rows satisfy relation (8) in Definition 3.6; therefore, the coefficient matrix $\boldsymbol{A}$ for the linear system (1) is strictly diagonally dominant.

The coefficient matrix $\boldsymbol{A}$ of the linear system (4) in Example 3.27 is not strictly diagonally dominant because

$$\begin{array}{ll} \text{In row 1:} & |-2| < |1| + |5| \\ \text{In row 2:} & |-8| > |4| + |1| \\ \text{In row 3:} & |1| < |4| + |-1|. \end{array}$$

Rows 1 and 3 do not satisfy relation (8) in Definition 3.6; therefore, the coefficient matrix $\boldsymbol{A}$ for the linear system (4) is not strictly diagonally dominant.

We now generalize the Jacobi and Gauss-Seidel iteration processes. Suppose that the given linear system is

$$\begin{array}{lllll} a_{11}x_1 + a_{12}x_2 & + \cdots + a_{1j}x_j + \cdots + & a_{1N}x_N & = b_1 \\ a_{21}x_1 + a_{22}x_2 & + \cdots + a_{2j}x_j + \cdots + & a_{2N}x_N & = b_2 \\ \vdots \qquad \vdots & \qquad \vdots & \vdots & \vdots \\ a_{j1}x_1 + a_{j2}x_2 & + \cdots + a_{jj}x_j + \cdots + & a_{jN}x_N & = b_j \\ \vdots \qquad \vdots & \qquad \vdots & \vdots & \vdots \\ a_{N1}x_1 + a_{N2}x_2 & + \cdots + a_{Nj}x_j + \cdots + & a_{NN}x_N & = b_N. \end{array} \tag{9}$$

Let the kth point be $\boldsymbol{P}_k = (x_1^{(k)}, x_2^{(k)}, \ldots, x_j^{(k)}, \ldots, x_N^{(k)})$; then the next point is $\boldsymbol{P}_{k+1} = (x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_j^{(k+1)}, \ldots, x_N^{(k+1)})$. The superscript (k) on the coordinates of $\boldsymbol{P}_k$ enables us to identify the coordinates that belong to this point. The iteration formulas use row j of (9) to solve for $x_j^{(k+1)}$ in terms of a linear combination of the previous values $x_1^{(k)}, x_2^{(k)}, \ldots, x_j^{(k)}, \ldots, x_N^{(k)}$:

Jacobi iteration:

$$x_j^{(k+1)} = \frac{b_j - a_{j1}x_1^{(k)} - \cdots - a_{jj-1}x_{j-1}^{(k)} - a_{jj+1}x_{j+1}^{(k)} - \cdots - a_{jN}x_N^{(k)}}{a_{jj}} \tag{10}$$

for $j = 1, 2, \ldots, N$.

Jacobi iteration uses all old coordinates to generate all new coordinates, whereas Gauss-Seidel iteration uses the new coordinates as they become available:

Gauss-Seidel iteration:

$$x_j^{(k+1)} = \frac{b_j - a_{j1}x_1^{(k+1)} - \cdots - a_{jj-1}x_{j-1}^{(k+1)} - a_{jj+1}x_{j+1}^{(k)} - \cdots - a_{jN}x_N^{(k)}}{a_{jj}} \tag{11}$$

for $j = 1, 2, \ldots, N$.

The following theorem gives a sufficient condition for Jacobi iteration to converge.

Theorem 3.15 (Jacobi Iteration). Suppose that $\boldsymbol{A}$ is a strictly diagonally dominant matrix. Then $\boldsymbol{AX} = \boldsymbol{B}$ has a unique solution $\boldsymbol{X} = \boldsymbol{P}$. Iteration using formula (10) will produce a sequence of vectors $\{\boldsymbol{P}_k\}$ that will converge to $\boldsymbol{P}$ for any choice of the starting vector $\boldsymbol{P}_0$.

Proof. The proof can be found in advanced texts on numerical analysis. •

It can be proved that the Gauss-Seidel method will also converge when the matrix $\boldsymbol{A}$ is strictly diagonally dominant. In many cases the Gauss-Seidel method will converge faster than the Jacobi method; hence it is usually preferred (compare Examples 3.26 and 3.28). It is important to understand the slight modification of formula (10) that has been made to obtain formula (11). In some cases the Jacobi method will converge even though the Gauss-Seidel method will not.

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