:

Polynomial Fitting

When the foregoing method is adapted to using the functions $\{f_j(x) = x^{j-1}\}$ and the index of summation ranges from $j = 1$ to $j = M + 1$, the function $f(x)$ will be a polynomial of degree M:

$$f(x) = c_1 + c_2 x + c_3 x^2 + \cdots + c_{M+1} x^M. \tag{26}$$

We now show how to find the ***least-squares parabola***, and the extension to a polynomial of higher degree is easily made and is left for the reader.

Theorem 5.3 (Least-Squares Parabola). Suppose that $\{(x_k, y_k)\}_{k=1}^{N}$ are N points, where the abscissas are distinct. The coefficients of the least-squares parabola

$$y = f(x) = Ax^2 + Bx + C \tag{27}$$

are the solution values A, B, and C of the linear system

$$\begin{aligned}
\left(\sum_{k=1}^{N} x_k^4\right) A + \left(\sum_{k=1}^{N} x_k^3\right) B + \left(\sum_{k=1}^{N} x_k^2\right) C &= \sum_{k=1}^{N} y_k x_k^2,\\
\left(\sum_{k=1}^{N} x_k^3\right) A + \left(\sum_{k=1}^{N} x_k^2\right) B + \left(\sum_{k=1}^{N} x_k\right) C &= \sum_{k=1}^{N} y_k x_k,\\
\left(\sum_{k=1}^{N} x_k^2\right) A + \left(\sum_{k=1}^{N} x_k\right) B + NC &= \sum_{k=1}^{N} y_k.
\end{aligned} \tag{28}$$

Table 5.7 Obtaining the Coefficients for the Least-Squares Parabola of Example 5.6

x_k	y_k	x_k^2	x_k^3	x_k^4	$x_k y_k$	$x_k^2 y_k$
−3	3	9	−27	81	−9	27
0	1	0	0	0	0	0
2	1	4	8	16	2	4
4	3	16	64	256	12	48
3	8	29	45	353	5	79

Proof. The coefficients A, B, and C will minimize the quantity:

$$E(A, B, C) = \sum_{k=1}^{N} (Ax_k^2 + Bx_k + C - y_k)^2. \tag{29}$$

The partial derivatives $\partial E/\partial A$, $\partial E/\partial B$, and $\partial E/\partial C$ must all be zero. This results in

$$\begin{aligned}
0 &= \frac{\partial E(A, B, C)}{\partial A} = 2\sum_{k=1}^{N} (Ax_k^2 + Bx_k + C - y_k)^1 (x_k^2),\\
0 &= \frac{\partial E(A, B, C)}{\partial B} = 2\sum_{k=1}^{N} (Ax_k^2 + Bx_k + C - y_k)^1 (x_k),\\
0 &= \frac{\partial E(A, B, C)}{\partial C} = 2\sum_{k=1}^{N} (Ax_k^2 + Bx_k + C - y_k)^1 (1).
\end{aligned} \tag{30}$$

Using the distributive property of addition, we can move the values A, B, and C outside the summations in (30) to obtain the normal equations that are given in (28). •

Example 5.6. Find the least-squares parabola for the four points (−3, 3), (0, 1), (2, 1), and (4, 3).

The entries in Table 5.7 are used to compute the summations required in the linear system (28).

The linear system (28) for finding A, B, and C becomes

$$\begin{aligned}
353A + 45B + 29C &= 79\\
45A + 29B + \ \ 3C &= \ \ 5\\
29A + \ \ 3B + \ \ 4C &= \ \ 8.
\end{aligned}$$

The solution to the linear system is $A = 585/3278$, $B = -631/3278$, and $C = 1394/1639$, and the desired parabola is (see Figure 5.8)

$$y = \frac{585}{3278}x^2 - \frac{631}{3278}x + \frac{1394}{1639} = 0.178462x^2 - 0.192495x + 0.850519. \qquad \blacksquare$$

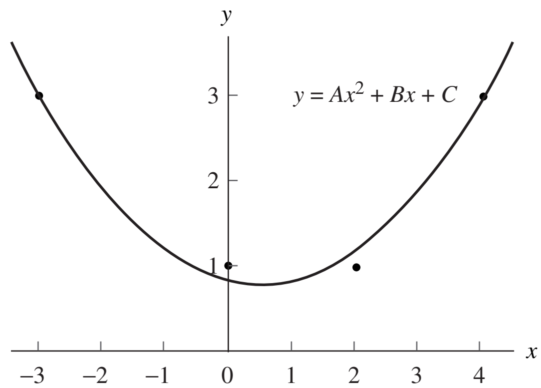

Figure 5.8 The least-squares parabola for Example 5.6.

Polynomial Wiggle

It is tempting to use a least-squares polynomial to fit data that are nonlinear. But if the data do not exhibit a polynomial nature, the resulting curve may exhibit large oscillations. This phenomenon, called ***polynomial wiggle***, becomes more pronounced with higher-degree polynomials. For this reason we seldom use a polynomial of degree 6 or above unless it is known that the true function we are working with is a polynomial.

For example, let $f(x) = 1.44/x^2 + 0.24x$ be used to generate the six data points (0.25, 23.1), (1.0, 1.68), (1.5, 1.0), (2.0, 0.84), (2.4, 0.826), and (5.0, 1.2576). The result of curve fitting with the least-squares polynomials

$$P_2(x) = 22.93 - 16.96x + 2.553x^2,$$
$$P_3(x) = 33.04 - 46.51x + 19.51x^2 - 2.296x^3,$$
$$P_4(x) = 39.92 - 80.93x + 58.39x^2 - 17.15x^3 + 1.680x^4,$$

and

$$P_5(x) = 46.02 - 118.1x + 119.4x^2 - 57.51x^3 + 13.03x^4 - 1.085x^5$$

is shown in Figure 5.9(a) through (d). Notice that $P_3(x)$, $P_4(x)$, and $P_5(x)$ exhibit a large wiggle in the interval [2, 5]. Even though $P_5(x)$ goes through the six points, it produces the worst fit. If we must fit a polynomial to these data, $P_2(x)$ should be the choice.

The following program uses the matrix $\boldsymbol{F}$ with entries $f_j(x) = x_k^{j-1}$ from equation (18).

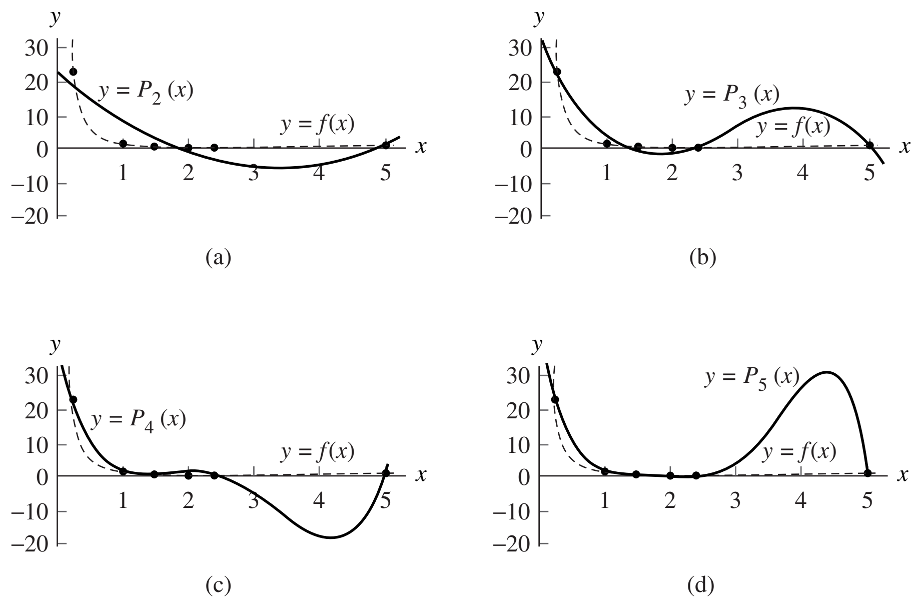

Figure 5.9 (a) Using $P_2(x)$ to fit data. (b) Using $P_3(x)$ to fit data. (c) Using $P_4(x)$ to fit data. (d) Using $P_5(x)$ to fit data.

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