

Triangular Factorization

We now discuss how to obtain the triangular factorization. If row interchanges are not necessary when using Gaussian elimination, the multipliers m_{ij} are the subdiagonal entries in L .

Example 3.21. Use Gaussian elimination to construct the triangular factorization of the matrix

$$A = \begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix}.$$

The matrix L will be constructed from an identity matrix placed at the left. For each row operation used to construct the upper-triangular matrix, the multipliers m_{ij} will be put in their proper places at the left. Start with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix}.$$

Row 1 is used to eliminate the elements of A in column 1 below a_{11} . The multiples $m_{21} = -0.5$ and $m_{31} = 0.25$ of row 1 are subtracted from rows 2 and 3, respectively. These multipliers are put in the matrix at the left and the result is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 1.25 & 6.25 \end{bmatrix}.$$

Row 2 is used to eliminate the elements in column 2 below the diagonal of the second factor of A in the above product. The multiple $m_{32} = -0.5$ of the second row is subtracted from row 3, and the multiplier is entered in the matrix at the left and we have the desired triangular factorization of A .

$$(8) \quad A = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix}. \quad \blacksquare$$

Theorem 3.10 (Direct Factorization $A = LU$: No Row Interchanges). Suppose that Gaussian elimination, without row interchanges, can be performed successfully to solve the general linear system $AX = B$. Then the matrix A can be factored as the product of a lower-triangular matrix L and an upper-triangular matrix U :

$$A = LU.$$

Furthermore, L can be constructed to have 1's on its diagonal and U will have nonzero diagonal elements. After finding L and U , the solution X is computed in two steps:

1. Solve $LY = B$ for Y using forward substitution.
2. Solve $UX = Y$ for X using back substitution.

Proof. We will show that, when the Gaussian elimination process is followed and B is stored in column $N + 1$ of the augmented matrix, the result after the upper-triangularization step is the equivalent upper-triangular system $UX = Y$. The matrices L , U , B , and Y will have the form

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{N1} & m_{N2} & m_{N3} & \cdots & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a_{1N+1}^{(1)} \\ a_{2N+1}^{(2)} \\ a_{3N+1}^{(3)} \\ \vdots \\ a_{NN+1}^{(N)} \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1N}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2N}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3N}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{NN}^{(N)} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} a_{1N+1}^{(1)} \\ a_{2N+1}^{(2)} \\ a_{3N+1}^{(3)} \\ \vdots \\ a_{NN+1}^{(N)} \end{bmatrix}.$$

Remark. To find just L and U , the $(N + 1)$ st column is not needed.

Step 1. Store the coefficients in the augmented matrix. The superscript on $a_{rc}^{(1)}$ means that this is the first time that a number is stored in location (r, c) .

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1N}^{(1)} & a_{1N+1}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2N}^{(1)} & a_{2N+1}^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3N}^{(1)} & a_{3N+1}^{(1)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{N1}^{(1)} & a_{N2}^{(1)} & a_{N3}^{(1)} & \cdots & a_{NN}^{(1)} & a_{NN+1}^{(1)} \end{array} \right]$$

Step 2. Eliminate x_1 in rows 2 through N and store the multiplier m_{r1} , used to eliminate x_1 in row r , in the matrix at location $(r, 1)$.

$$\begin{aligned} &\text{for } r = 2 : N \\ &\quad m_{r1} = a_{r1}^{(1)} / a_{11}^{(1)}; \\ &\quad a_{r1} = m_{r1}; \\ &\text{for } c = 2 : N + 1 \end{aligned}$$

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 $a_{rc}^{(2)} = a_{rc}^{(1)} - m_{r1} * a_{1c}^{(1)};$ 
end
end

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The new elements are written $a_{rc}^{(2)}$ to indicate that this is the second time that a number has been stored in the matrix at location (r, c) . The result after step 2 is

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1N}^{(1)} & a_{1N+1}^{(1)} \\ m_{21} & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2N}^{(2)} & a_{2N+1}^{(2)} \\ m_{31} & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3N}^{(2)} & a_{3N+1}^{(2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ m_{N1} & a_{N2}^{(2)} & a_{N3}^{(2)} & \cdots & a_{NN}^{(2)} & a_{NN+1}^{(2)} \end{array} \right]$$

Step 3. Eliminate x_2 in rows 3 through N and store the multiplier m_{r2} , used to eliminate x_2 in row r , in the matrix at location $(r, 2)$.

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for  $r = 3 : N$ 
 $m_{r2} = a_{r2}^{(2)} / a_{22}^{(2)};$ 
 $a_{r2} = m_{r2};$ 
for  $c = 3 : N + 1$ 
 $a_{rc}^{(3)} = a_{rc}^{(2)} - m_{r2} * a_{2c}^{(2)};$ 
end
end

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The new elements are written $a_{rc}^{(3)}$ to indicate that this is the third time that a number has been stored in the matrix at the location (r, c) .

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1N}^{(1)} & a_{1N+1}^{(1)} \\ m_{21} & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2N}^{(2)} & a_{2N+1}^{(2)} \\ m_{31} & m_{32} & a_{33}^{(3)} & \cdots & a_{3N}^{(3)} & a_{3N+1}^{(3)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ m_{N1} & m_{N2} & a_{N3}^{(3)} & \cdots & a_{NN}^{(3)} & a_{NN+1}^{(3)} \end{array} \right]$$

Step $p + 1$. This is the general step. Eliminate x_p in rows $p + 1$ through N and store the multipliers at the location (r, p) .

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for  $r = p + 1 : N$ 
 $m_{rp} = a_{rp}^{(p)} / a_{pp}^{(p)};$ 

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 $a_{rp} = m_{rp};$ 
for  $c = p + 1 : N + 1$ 
   $a_{rc}^{(p+1)} = a_{rc}^{(p)} - m_{rp} * a_{pc}^{(p)};$ 
end
end

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The final result after x_{N-1} has been eliminated from row N is

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1N}^{(1)} & a_{1N+1}^{(1)} \\ m_{21} & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2N}^{(2)} & a_{2N+1}^{(2)} \\ m_{31} & m_{32} & a_{33}^{(3)} & \cdots & a_{3N}^{(3)} & a_{3N+1}^{(3)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ m_{N1} & m_{N2} & m_{N3} & \cdots & a_{NN}^{(N)} & a_{NN+1}^{(N)} \end{array} \right]$$

The upper-triangular process is now complete. Notice that one array is used to store the elements of both L and U . The 1's of L are not stored, nor are the 0's of L and U that lie above and below the diagonal, respectively. Only the essential coefficients needed to reconstruct L and U are stored!

We must now verify that the product $LU = A$. Suppose that $D = LU$ and consider the case when $r \leq c$. Then d_{rc} is

$$(9) \quad d_{rc} = m_{r1}a_{1c}^{(1)} + m_{r2}a_{2c}^{(2)} + \cdots + m_{rr-1}a_{r-1c}^{(r-1)} + a_{rc}^{(r)}.$$

Using the replacement equations in steps 1 through $p + 1 = r$, we obtain the following substitutions:

$$(10) \quad \begin{aligned} m_{r1}a_{1c}^{(1)} &= a_{rc}^{(1)} - a_{rc}^{(2)}, \\ m_{r2}a_{2c}^{(2)} &= a_{rc}^{(2)} - a_{rc}^{(3)}, \\ &\vdots \\ m_{rr-1}a_{r-1c}^{(r-1)} &= a_{rc}^{(r-1)} - a_{rc}^{(r)}. \end{aligned}$$

When the substitutions in (10) are used in (9), the result is

$$d_{rc} = a_{rc}^{(1)} - a_{rc}^{(2)} + a_{rc}^{(2)} - a_{rc}^{(3)} + \cdots + a_{rc}^{(r-1)} - a_{rc}^{(r)} + a_{rc}^{(r)} = a_{rc}^{(1)}.$$

The other case, $r > c$, is similar to prove. •

Computational Complexity

The process for triangularizing is the same for both the Gaussian elimination and triangular factorization methods. We can count the operations if we look at the first N

columns of the augmented matrix in Theorem 3.10. The outer loop of step $p + 1$ requires $N - p = N - (p + 1) + 1$ divisions to compute the multipliers m_{rp} . Inside the loops, but for the first N columns only, a total of $(N - p)(N - p)$ multiplications and the same number of subtractions are required to compute the new row elements $a_{rc}^{(p+1)}$. This process is carried out for $p = 1, 2, \dots, N - 1$. Thus the triangular factorization portion of $A = LU$ requires

$$(11) \quad \sum_{p=1}^{N-1} (N - p)(N - p + 1) = \frac{N^3 - N}{3} \quad \text{multiplications and divisions}$$

and

$$(12) \quad \sum_{p=1}^{N-1} (N - p)(N - p) = \frac{2N^3 - 3N^2 + N}{6} \quad \text{subtractions.}$$

To establish (11), we use the summation formulas

$$\sum_{k=1}^M k = \frac{M(M + 1)}{2} \quad \text{and} \quad \sum_{k=1}^M k^2 = \frac{M(M + 1)(2M + 1)}{6}.$$

Using the change of variables $k = N - p$, we rewrite (11) as

$$\begin{aligned} \sum_{p=1}^{N-1} (N - p)(N - p + 1) &= \sum_{p=1}^{N-1} (N - p) + \sum_{p=1}^{N-1} (N - p)^2 \\ &= \sum_{k=1}^{N-1} k + \sum_{k=1}^{N-1} k^2 \\ &= \frac{(N - 1)N}{2} + \frac{(N - 1)(N)(2N - 1)}{6} \\ &= \frac{N^3 - N}{3}. \end{aligned}$$

Once the triangular factorization $A = LU$ has been obtained, the solution to the lower-triangular system $LY = B$ will require $0 + 1 + \dots + N - 1 = (N^2 - N)/2$ multiplications and subtractions; no divisions are required because the diagonal elements of L are 1's. Then the solution of the upper-triangular system $UX = Y$ requires $1 + 2 + \dots + N = (N^2 + N)/2$ multiplications and divisions and $(N^2 - N)/2$ subtractions. Therefore, finding the solution to $LUX = B$ requires

$$N^2 \text{ multiplications and divisions, and } N^2 - N \text{ subtractions.}$$

We see that the bulk of the calculations lies in the triangularization portion of the solution. If the linear system is to be solved many times, with the same coefficient

matrix A but with different column matrices B , it is not necessary to triangularize the matrix each time if the factors are saved. This is the reason the triangular factorization method is usually chosen over the elimination method. However, if only one linear system is solved, the two methods are the same, except that the triangular factorization method stores the multipliers.

Permutation Matrices

The $A = LU$ factorization in Theorem 3.10 assumes that there are no row interchanges. It is possible that a nonsingular matrix A cannot be factored directly as $A = LU$.

Example 3.22. Show that the following matrix cannot be factored directly as $A = LU$:

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix}.$$

Suppose that A has a direct factorization LU ; then

$$(13) \quad \begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

The matrices L and U on the right-hand side of (13) can be multiplied and each element of the product compared with the corresponding element of the matrix A . In the first column, $1 = 1u_{11}$, then $4 = m_{21}u_{11} = m_{21}$, and finally, $-2 = m_{31}u_{11} = m_{31}$. In the second column, $2 = 1u_{12}$, then $8 = m_{21}u_{12} = (4)(2) + u_{22}$ implies that $u_{22} = 0$; and finally, $3 = m_{31}u_{12} + m_{32}u_{22} = (-2)(2) + m_{32}(0) = -4$, which is a contradiction. Therefore, A does not have a LU factorization. ■

A permutation of the first N positive integers $1, 2, \dots, N$ is an arrangement k_1, k_2, \dots, k_N of these integers in a definite order. For example, $1, 4, 2, 3, 5$ is a permutation of the five integers $1, 2, 3, 4, 5$. The standard base vectors $E_i = [0 \ 0 \ \cdots \ 0 \ 1_i \ 0 \ \cdots \ 0]$, for $i = 1, 2, \dots, N$, are used in the next definition.

Definition 3.5. An $N \times N$ *permutation matrix* P is a matrix with precisely one entry whose value is 1 in each column and row, and all of whose other entries are 0. The rows of P are a permutation of the rows of the identity matrix and can be written as

$$(14) \quad P = [E'_{k_1} \ E'_{k_2} \ \cdots \ E'_{k_N}]'$$

The elements of $P = [p_{ij}]$ have the form

$$p_{ij} = \begin{cases} 1 & j = k_i, \\ 0 & \text{otherwise.} \end{cases}$$

For example, the following 4×4 matrix is a permutation matrix,

$$(15) \quad \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [\mathbf{E}'_2 \quad \mathbf{E}'_1 \quad \mathbf{E}'_4 \quad \mathbf{E}'_3]'. \quad \blacktriangle$$

Theorem 3.11. Suppose that $\mathbf{P} = [\mathbf{E}'_{k_1} \quad \mathbf{E}'_{k_2} \quad \cdots \quad \mathbf{E}'_{k_N}]'$ is a permutation matrix. The product \mathbf{PA} is a new matrix whose rows consist of the rows of \mathbf{A} rearranged in the order $\text{row}_{k_1} \mathbf{A}, \text{row}_{k_2} \mathbf{A}, \dots, \text{row}_{k_N} \mathbf{A}$.

Example 3.23. Let \mathbf{A} be a 4×4 matrix and let \mathbf{P} be the permutation matrix given in (15); then \mathbf{PA} is the matrix whose rows consist of the rows of \mathbf{A} rearranged in the order $\text{row}_2 \mathbf{A}, \text{row}_1 \mathbf{A}, \text{row}_4 \mathbf{A}, \text{row}_3 \mathbf{A}$.

Computing the product, we have

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}. \quad \blacksquare$$

Theorem 3.12. If \mathbf{P} is a permutation matrix, then it is nonsingular and $\mathbf{P}^{-1} = \mathbf{P}'$.

Theorem 3.13. If \mathbf{A} is a nonsingular matrix, then there exists a permutation matrix \mathbf{P} so that \mathbf{PA} has a triangular factorization

$$(16) \quad \mathbf{PA} = \mathbf{LU}.$$

The proofs can be found in advanced linear algebra texts.

Example 3.24. If rows 2 and 3 of the matrix in Example 3.22 are interchanged, then the resulting matrix \mathbf{PA} has a triangular factorization.

The permutation matrix that switches rows 2 and 3 is $\mathbf{P} = [\mathbf{E}'_1 \quad \mathbf{E}'_3 \quad \mathbf{E}'_2]'$. Computing the product \mathbf{PA} , we obtain

$$\mathbf{PA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ -2 & 3 & 5 \\ 4 & 8 & -1 \end{bmatrix}.$$

Now Gaussian elimination without row interchanges can be used:

$$\begin{array}{l} \text{pivot} \rightarrow \\ m_{21} = -2 \\ m_{31} = 4 \end{array} \begin{bmatrix} 1 & 2 & 6 \\ -2 & 3 & 5 \\ 4 & 8 & -1 \end{bmatrix}.$$

After x_2 has been eliminated from column 2, row 3, we have

$$\begin{array}{l} \text{pivot} \rightarrow \\ m_{32} = 0 \end{array} \begin{bmatrix} 1 & 2 & 6 \\ 0 & 7 & 17 \\ 0 & 0 & -25 \end{bmatrix} = \mathbf{U}. \quad \blacksquare$$

Extending the Gaussian Elimination Process

The following theorem is an extension of Theorem 3.10, which includes the cases when row interchanges are required. Thus triangular factorization can be used to find the solution to any linear system $AX = B$, where A is nonsingular.

Theorem 3.14 (Indirect Factorization: $PA = LU$). Let A be a given $N \times N$ matrix. Assume that Gaussian elimination can be performed successfully to solve the general linear system $AX = B$, but that row interchanges are required. Then there exists a permutation matrix P so that the product PA can be factored as the product of a lower-triangular matrix L and an upper-triangular matrix U :

$$PA = LU.$$

Furthermore, L can be constructed to have 1's on its main diagonal and U will have nonzero diagonal elements. The solution X is found in four steps:

1. Construct the matrices L , U , and P .
2. Compute the column vector PB .
3. Solve $LY = PB$ for Y using forward substitution.
4. Solve $UX = Y$ for X using back substitution.

Remark. Suppose that $AX = B$ is to be solved for a fixed matrix A and several different column matrices B . Then step 1 is performed only once and steps 2 through 4 are used to find the solution X that corresponds to B . Steps 2 through 4 are a computationally efficient way to construct the solution X and require $O(N^2)$ operations instead of the $O(N^3)$ operations required by Gaussian elimination.

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