Milne-Simpson Method

Another popular predictor-corrector scheme is known as the Milne-Simpson method. Its predictor is based on integration of $f(t, y(t))$ over the interval $[t_{k-3}, t_{k+1}]$:

$$y(t_{k+1}) = y(t_{k-3}) + \int_{t_{k-3}}^{t_{k+1}} f(t, y(t))\,dt. \tag{10}$$

The predictor uses the Lagrange polynomial approximation for $f(t, y(t))$ based on the points (t_{k-3}, f_{k-3}), (t_{k-2}, f_{k-2}), (t_{k-1}, f_{k-1}), and (t_k, f_k). It is integrated over the interval $[t_{k-3}, t_{k+1}]$. This produces the Milne predictor:

$$p_{k+1} = y_{k-3} + \frac{4h}{3}(2f_{k-2} - f_{k-1} + 2f_k). \tag{11}$$

The corrector is developed similarly. The value p_{k+1} can now be used. A second Lagrange polynomial for $f(t, y(t))$ is constructed, which is based on the points (t_{k-1}, f_{k-1}), (t_k, f_k), and the new point $(t_{k+1}, f_{k+1}) = (t_{k+1}, f(t_{k+1}, p_{k+1}))$. The polynomial is integrated over $[t_{k-1}, t_{k+1}]$, and the result is the familiar Simpson's rule:

$$y_{k+1} = y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_k + f_{k+1}). \tag{12}$$

Error Estimation and Correction

The error terms for the numerical integration formulas used to obtain both the predictor and corrector are of the order $\boldsymbol{O}(h^5)$. The L.T.E. for the formulas in (11) and (12) are

$$y(t_{k+1}) - p_{k+1} = \frac{28}{90} y^{(5)}(c_{k+1})h^5 \qquad \text{(L.T.E. for the predictor)}, \tag{13}$$

$$y(t_{k+1}) - y_{k+1} = \frac{-1}{90} y^{(5)}(d_{k+1})h^5 \qquad \text{(L.T.E. for the corrector)}. \tag{14}$$

Suppose that h is small enough so that $y^{(5)}(t)$ is nearly constant over the interval $[t_{k-3}, t_{k+1}]$. Then the terms involving the fifth derivative can be eliminated in (13) and (14) and the result is

$$y(t_{k+1}) - p_{k+1} \approx \frac{28}{29}(y_{k+1} - p_{k+1}). \tag{15}$$

Formula (15) gives an error estimate for the predictor that is based on the two computed values p_{k+1} and y_{k+1} and does not use $y^{(5)}(t)$. It can be used to improve the predicted value. Under the assumption that the difference between the predicted and corrected values at each step changes slowly, we can substitute p_k and y_k for p_{k+1} and y_{k+1} in (15) and get the following modifier:

$$m_{k+1} = p_{k+1} + 28\frac{y_k - p_k}{29}. \tag{16}$$

This modified value is used in place of p_{k+1} in the correction step, and equation (12) becomes

$$y_{k+1} = y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_k + f(t_{k+1}, m_{k+1})). \tag{17}$$

Therefore, the improved (modified) Milne-Simpson method is

$$\begin{aligned}
p_{k+1} &= y_{k-3} + \frac{4h}{3}(2f_{k-2} - f_{k-1} + 2f_k) && \text{(predictor)}\\
m_{k+1} &= p_{k+1} + 28\frac{y_k - p_k}{29} && \text{(modifier)}\\
f_{k+1} &= f(t_{k+1}, m_{k+1}) &&\\
y_{k+1} &= y_{k-1} + \frac{h}{3}(f_{k-1} + 4f_k + f_{k+1}) && \text{(corrector).}
\end{aligned} \tag{18}$$

Hamming's method is another important method. We shall omit its derivation, but furnish a program at the end of the section. As a final precaution we mention that all the predictor-corrector methods have stability problems. Stability is an advanced topic and the serious reader should research this subject.

Example 9.13. Use the Adams-Bashforth-Moulton, Milne-Simpson, and Hamming methods with $h = \frac{1}{8}$ and compute approximations for the solution of the I.V.P.

$$y' = \frac{t - y}{2}, \qquad y(0) = 1 \qquad \text{over } [0, 3].$$

A Runge-Kutta method was used to obtain the starting values

$$y_1 = 0.94323919, \qquad y_2 = 0.89749071, \qquad \text{and} \qquad y_3 = 0.86208736.$$

Then a computer implementation of Programs 9.6 through 9.8 produced the values in Table 9.12. The error for each entry in the table is given as a multiple of 10^{-8}. In all entries there are at least six digits of accuracy. In this example, the best answers were produced by Hamming's method. ∎

Table 9.12 Comparison of the Adams-Bashforth-Moulton, Milne-Simpson, and Hamming Methods for Solving $y' = (t - y)/2$, $y(0) = 1$

k	Adams-Bashforth-Moulton	Error	Milne-Simpson	Error	Hamming's method	Error
0.0	1.00000000	$0E-8$	1.00000000	$0E-8$	1.00000000	$0E-8$
0.5	0.83640227	$8E-8$	0.83640231	$4E-8$	0.83640234	$1E-8$
0.625	0.81984673	$16E-8$	0.81984687	$2E-8$	0.81984688	$1E-8$
0.75	0.81186762	$22E-8$	0.81186778	$6E-8$	0.81186783	$1E-8$
0.875	0.81194530	$28E-8$	0.81194555	$3E-8$	0.81194558	$0E-8$
1.0	0.81959166	$32E-8$	0.81959190	$8E-8$	0.81959198	$0E-8$
1.5	0.91709920	$46E-8$	0.91709957	$9E-8$	0.91709967	$-1E-8$
2.0	1.10363781	$51E-8$	1.10363822	$10E-8$	1.10363834	$-2E-8$
2.5	1.35951387	$52E-8$	1.35951429	$10E-8$	1.35951441	$-2E-8$
2.625	1.43243853	$52E-8$	1.43243899	$6E-8$	1.43243907	$-2E-8$
2.75	1.50851827	$52E-8$	1.50851869	$10E-8$	1.50851881	$-2E-8$
2.875	1.58756195	$51E-8$	1.58756240	$6E-8$	1.58756248	$-2E-8$
3.0	1.66938998	$50E-8$	1.66939038	$10E-8$	1.66939050	$-2E-8$

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