Richardson's Extrapolation

In this section we emphasize the relationship between formulas (3) and (10). Let $f_k = f(x_k) = f(x_0 + kh)$, and use the notation $D_0(h)$ and $D_0(2h)$ to denote the approximations to $f'(x_0)$ that are obtained from (3) with step sizes h and $2h$, respectively:

$$f'(x_0) \approx D_0(h) + Ch^2 \tag{27}$$

and

$$f'(x_0) \approx D_0(2h) + 4Ch^2. \tag{28}$$

If we multiply relation (27) by 4 and subtract relation (28) from this product, then the terms involving C cancel and the result is

$$3f'(x_0) \approx 4D_0(h) - D_0(2h) = \frac{4(f_1 - f_{-1})}{2h} - \frac{f_2 - f_{-2}}{4h}. \tag{29}$$

Next solve for $f'(x_0)$ in (29) and get

$$f'(x_0) \approx \frac{4D_0(h) - D_0(2h)}{3} = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h}. \tag{30}$$

The last expression in (30) is the central-difference formula (10).

Example 6.3. Let $f(x) = \cos(x)$. Use (27) and (28) with $h = 0.01$, and show how the linear combination $(4D_0(h) - D_0(2h))/3$ in (30) can be used to obtain the approximation to $f'(0.8)$ given in (10). Carry nine decimal places in all the calculations.

Use (27) and (28) with $h = 0.01$ to get

$$D_0(h) \approx \frac{f(0.81) - f(0.79)}{0.02} \approx \frac{0.689498433 - 0.703845316}{0.02} \approx -0.717344150$$

and

$$D_0(2h) \approx \frac{f(0.82) - f(0.78)}{0.04} \approx \frac{0.682221207 - 0.710913538}{0.04} \approx -0.717308275.$$

Now the linear combination in (30) is computed:

$$f'(0.8) \approx \frac{4D_0(h) - D_0(2h)}{3} \approx \frac{4(-0.717344150) - (-0.717308275)}{3} \approx -0.717356108.$$

This is exactly the same as the solution in Example 6.2 that used (10) directly to approximate $f'(0.8)$. ■

The method of obtaining a formula for $f'(x_0)$ of higher order from a formula of lower order is called ***extrapolation***. The proof requires that the error term for (3) can be expanded in a series containing only even powers of h. We have already seen how to use step sizes h and $2h$ to remove the term involving h^2. To see how h^4 is removed, let $D_1(h)$ and $D_1(2h)$ denote the approximations to $f'(x_0)$ of order $\boldsymbol{O}(h^4)$ obtained with formula (16) using step sizes h and $2h$, respectively. Then

$$(31) \qquad f'(x_0) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{24h} + \frac{h^4 f^{(5)}(c_1)}{30} \approx D_1(h) + Ch^4$$

and

$$(32) \qquad f'(x_0) = \frac{-f_4 + 8f_2 - 8f_{-2} + f_{-4}}{12h} + \frac{16h^4 f^{(5)}(c_2)}{30} \approx D_1(2h) + 16Ch^4.$$

Suppose that $f^{(5)}(x)$ has one sign and does not change too rapidly; then the assumption that $f^{(5)}(c_1) \approx f^{(5)}(c_2)$ can be used to eliminate the terms involving h^4 in (31) and (32), and the result is

$$(33) \qquad f'(x_0) \approx \frac{16D_1(h) - D_1(2h)}{15}.$$

The general pattern for improving calculations is stated in the next result.

Theorem 6.3 (Richardson's Extrapolation). Suppose that two approximations of order $\boldsymbol{O}(h^{2k})$ for $f'(x_0)$ are $D_{k-1}(h)$ and $D_{k-1}(2h)$ and that they satisfy

$$f'(x_0) = D_{k-1}(h) + c_1 h^{2k} + c_2 h^{2k+2} + \cdots \tag{34}$$

and

$$f'(x_0) = D_{k-1}(2h) + 4^k c_1 h^{2k} + 4^{k+1} c_2 h^{2k+2} + \cdots . \tag{35}$$

Then an improved approximation has the form

$$f'(x_0) = D_k(h) + \boldsymbol{O}(h^{2k+2}) = \frac{4^k D_{k-1}(h) - D_{k-1}(2h)}{4^k - 1} + \boldsymbol{O}(h^{2k+2}). \tag{36}$$

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