7.3 Recursive Rules and Romberg Integration

In this section we show how to compute Simpson approximations with a special linear combination of trapezoidal rules. The approximation will have greater accuracy if one uses a larger number of subintervals. How many should we choose? The sequential process helps answer this question by trying two subintervals, four subintervals, and so on, until the desired accuracy is obtained. First, a sequence $\{T(J)\}$ of trapezoidal rule approximations must be generated. As the number of subintervals is doubled, the number of function values is roughly doubled, because the function must be evaluated at all the previous points and at the midpoints of the previous subintervals (see Figure 7.8). Theorem 7.4 explains how to eliminate redundant function evaluations and additions.

Theorem 7.4 (Successive Trapezoidal Rules). Suppose that $J \geq 1$ and the points $\{x_k = a + kh\}$ subdivide $[a, b]$ into $2^J = 2M$ subintervals of equal width $h = (b - a)/2^J$. The trapezoidal rules $T(f, h)$ and $T(f, 2h)$ obey the relationship

$$T(f, h) = \frac{T(f, 2h)}{2} + h \sum_{k=1}^{M} f(x_{2k-1}). \tag{1}$$

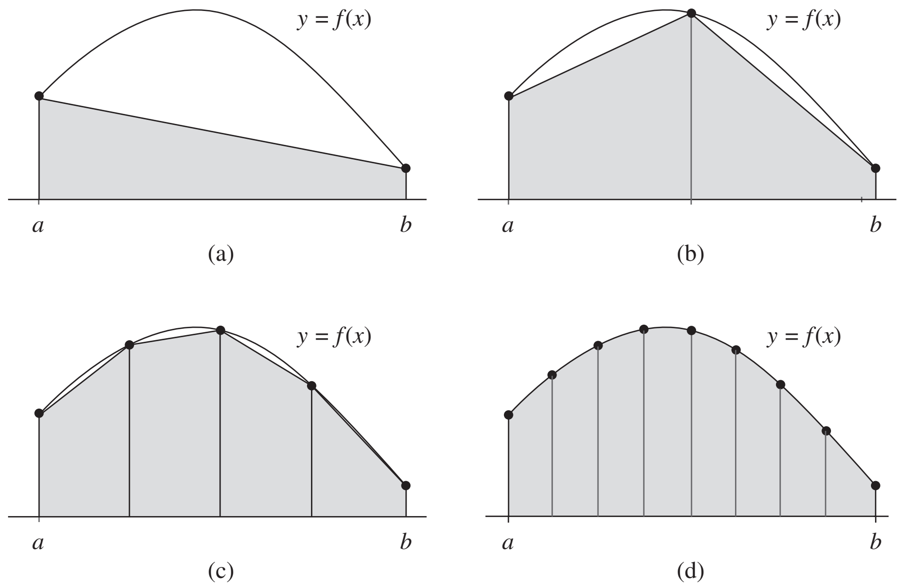

Figure 7.8 (a) $T(0)$ is the area under $2^0 = 1$ trapezoid. (b) $T(1)$ is the area under $2^1 = 2$ trapezoids. (c) $T(2)$ is the area under $2^2 = 4$ trapezoids. (d) $T(3)$ is the area under $2^3 = 8$ trapezoids.

Definition 7.3 (Sequence of Trapezoidal Rules). Define $T(0) = (h/2)(f(a) + f(b))$, which is the trapezoidal rule with step size $h = b - a$. Then for each $J \geq 1$ define $T(J) = T(f, h)$, where $T(f, h)$ is the trapezoidal rule with step size $h = (b - a)/2^J$. ▲

Corollary 7.4 (Recursive Trapezoidal Rule). Start with $T(0) = (h/2)(f(a) + f(b))$. Then a sequence of trapezoidal rules $\{T(J)\}$ is generated by the recursive formula

$$T(J) = \frac{T(J-1)}{2} + h\sum_{k=1}^{M} f(x_{2k-1}) \qquad \text{for } J = 1,\ 2,\ \ldots, \tag{2}$$

where $h = (b - a)/2^J$ and $\{x_k = a + kh\}$.

Proof. For the even nodes $x_0 < x_2 < \cdots < x_{2M-2} < x_{2M}$, we use the trapezoidal rule with step size $2h$:

$$T(J-1) = \frac{2h}{2}(f_0 + 2f_2 + 2f_4 + \cdots + 2f_{2M-4} + 2f_{2M-2} + f_{2M}). \tag{3}$$

For all of the nodes $x_0 < x_1 < x_2 < \cdots < x_{2M-1} < x_{2M}$, we use the trapezoidal rule with step size h:

$$T(J) = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{2M-2} + 2f_{2M-1} + f_{2M}). \tag{4}$$

Collecting the even and odd subscripts in (4) yields

$$T(J) = \frac{h}{2}(f_0 + 2f_2 + \cdots + 2f_{2M-2} + f_{2M}) + h\sum_{k=1}^{M} f_{2k-1}. \tag{5}$$

Substituting (3) into (5) results in $T(J) = T(J-1)/2 + h\sum_{k=1}^{M} f_{2k-1}$, and the proof of the theorem is complete. •

Example 7.11. Use the sequential trapezoidal rule to compute the approximations $T(0)$, $T(1)$, $T(2)$, and $T(3)$ for the integral $\int_1^5 dx/x = \ln(5) - \ln(1) = 1.609437912$.

Table 7.4 shows the nine values required to compute $T(3)$ and the midpoints required to compute $T(1)$, $T(2)$, and $T(3)$. Details for obtaining the results are as follows:

$$\text{When } h = 4: \quad T(0) = \frac{4}{2}(1.000000 + 0.200000) = 2.400000.$$

$$\begin{aligned}
\text{When } h = 2: \quad T(1) &= \frac{T(0)}{2} + 2(0.333333) \\
&= 1.200000 + 0.666666 = 1.866666.
\end{aligned}$$

$$\begin{aligned}
\text{When } h = 1: \quad T(2) &= \frac{T(1)}{2} + 1(0.500000 + 0.250000) \\
&= 0.933333 + 0.750000 = 1.683333.
\end{aligned}$$

$$\begin{aligned}
\text{When } h = \frac{1}{2}: \quad T(3) &= \frac{T(2)}{2} + \frac{1}{2}(0.666667 + 0.400000 \\
&\qquad + 0.285714 + 0.222222) \\
&= 0.841667 + 0.787302 = 1.628968.
\end{aligned}$$

■

Our next result shows an important relationship between the trapezoidal rule and Simpson's rule. When the trapezoidal rule is computed using step sizes $2h$ and h, the result is $T(f, 2h)$ and $T(f, h)$, respectively. These values are combined to obtain Simpson's rule:

$$S(f, h) = \frac{4T(f, h) - T(f, 2h)}{3}. \tag{6}$$

Theorem 7.5 (Recursive Simpson Rules). Suppose that $\{T(J)\}$ is the sequence of trapezoidal rules generated by Corollary 7.4. If $J \geq 1$ and $S(J)$ is Simpson's rule for 2^J subintervals of $[a, b]$, then $S(J)$ and the trapezoidal rules $T(J-1)$ and $T(J)$ obey the relationship

$$S(J) = \frac{4T(J) - T(J-1)}{3} \quad \text{for } J = 1, 2, \ldots. \tag{7}$$

Table 7.4 The Nine Points Used to Compute $T(3)$ and the Midpoints Required to Compute $T(1)$, $T(2)$, and $T(3)$

x	$f(x) = \frac{1}{x}$	Endpoints for computing $T(0)$	Midpoints for computing $T(1)$	Midpoints for computing $T(2)$	Midpoints for computing $T(3)$
1.0	1.000000	1.000000			
1.5	0.666667				0.666667
2.0	0.500000			0.500000	
2.5	0.400000				0.400000
3.0	0.333333		0.333333		
3.5	0.285714				0.285714
4.0	0.250000			0.250000	
4.5	0.222222				0.222222
5.0	0.200000	0.200000			

Proof. The trapezoidal rule $T(J)$ with step size h yields the approximation

$$\int_a^b f(x)\,dx \approx \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{2M-2} + 2f_{2M-1} + f_{2M}) = T(J). \tag{8}$$

The trapezoidal rule $T(J-1)$ with step size $2h$ produces

$$\int_a^b f(x)\,dx \approx h(f_0 + 2f_2 + \cdots + 2f_{2M-2} + f_{2M}) = T(J-1). \tag{9}$$

Multiplying relation (8) by 4 yields

$$4\int_a^b f(x)\,dx \approx h(2f_0 + 4f_1 + 4f_2 + \cdots + 4f_{2M-2} + 4f_{2M-1} + 2f_M) = 4T(J). \tag{10}$$

Now subtract (9) from (10) and the result is

$$3\int_a^b f(x)\,dx \approx h(f_0 + 4f_1 + 2f_2 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M}) = 4T(J) - T(J-1). \tag{11}$$

This can be rearranged to obtain

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M}) = \frac{4T(J) - T(J-1)}{3}. \tag{12}$$

The middle term in (12) is Simpson's rule $S(J) = S(f, h)$ and hence the theorem is proved. •

Example 7.12. Use the sequential Simpson rule to compute the approximations $S(1)$, $S(2)$, and $S(3)$ for the integral of Example 7.11.

Using the results of Example 7.11 and formula (7) with $J = 1$, 2, and 3, we compute

$$S(1) = \frac{4T(1) - T(0)}{3} = \frac{4(1.866666) - 2.400000}{3} = 1.688888,$$
$$S(2) = \frac{4T(2) - T(1)}{3} = \frac{4(1.683333) - 1.866666}{3} = 1.622222,$$
$$S(3) = \frac{4T(3) - T(2)}{3} = \frac{4(1.628968) - 1.683333}{3} = 1.610846. \quad \blacksquare$$

In Section 7.1 the formula for Boole's rule was given in Theorem 7.1. It was obtained by integrating the Lagrange polynomial of degree 4 based on the nodes x_0, x_1, x_2, x_3, and x_4. An alternative method for establishing Boole's rule is mentioned in the exercises. When it is applied M times over $4M$ equally spaced subintervals of $[a, b]$ of step size $h = (b - a)/(4M)$, we call it the ***composite Boole rule***:

$$(13) \qquad B(f, h) = \frac{2h}{45} \sum_{k=1}^{M} (7 f_{4k-4} + 32 f_{4k-3} + 12 f_{4k-2} + 32 f_{4k-1} + 7 f_{4k}).$$

The next result gives the relationship between the sequential Boole and Simpson rules.

Theorem 7.6 (Recursive Boole Rules). Suppose that $\{S(J)\}$ is the sequence of Simpson's rules generated by Theorem 7.5. If $J \geq 2$ and $B(J)$ is Boole's rule for 2^J subintervals of $[a, b]$, then $B(J)$ and Simpson's rules $S(J - 1)$ and $S(J)$ obey the relationship

$$(14) \qquad B(J) = \frac{16S(J) - S(J-1)}{15} \qquad \text{for } J = 2, 3, \ldots.$$

Proof. The proof is left as an exercise for the reader. •

Example 7.13. Use the sequential Boole rule to compute the approximations $B(2)$ and $B(3)$ for the integral of Example 7.11.

Using the results of Example 7.12 and formula (14) with $J = 2$ and 3, we compute

$$B(2) = \frac{16S(2) - S(1)}{15} = \frac{16(1.622222) - 1.688888}{15} = 1.617778,$$
$$B(3) = \frac{16S(3) - S(2)}{15} = \frac{16(1.610846) - 1.622222}{15} = 1.610088. \quad \blacksquare$$

The reader may wonder what we are leading up to. We will now show that formulas (7) and (14) are special cases of the process of Romberg integration. Let us announce that the next level of approximation for the integral of Example 7.11 is

$$\frac{64B(3) - B(2)}{63} = \frac{64(1.610088) - 1.617778}{63} = 1.609490,$$

and this answer gives an accuracy of five decimal places.

Romberg Integration

In Section 7.2 we saw that the error terms $E_T(f, h)$ and $E_S(f, h)$ for the composite trapezoidal rule and composite Simpson rule are of order $\boldsymbol{O}(h^2)$ and $\boldsymbol{O}(h^4)$, respectively. It is not difficult to show that the error term $E_B(f, h)$ for the composite Boole rule is of the order $\boldsymbol{O}(h^6)$. Thus we have the pattern

$$\int_a^b f(x)\,dx = T(f, h) + \boldsymbol{O}(h^2), \tag{15}$$

$$\int_a^b f(x)\,dx = S(f, h) + \boldsymbol{O}(h^4), \tag{16}$$

$$\int_a^b f(x)\,dx = B(f, h) + \boldsymbol{O}(h^6). \tag{17}$$

The pattern for the remainders in (15) through (17) is extended in the following sense. Suppose that an approximation rule is used with step sizes h and $2h$; then an algebraic manipulation of the two answers is used to produce an improved answer. Each successive level of improvement increases the order of the error term from $\boldsymbol{O}(h^{2N})$ to $\boldsymbol{O}(h^{2N+2})$. This process, called ***Romberg integration***, has its strengths and weaknesses.

The Newton-Cotes rules are seldom used past Boole's rule. This is because the nine-point Newton-Cotes quadrature rule involves negative weights, and all the rules past the 10-point rule involve negative weights. This could introduce loss of significance error due to round off. The Romberg method has the advantages that all the weights are positive and the equally spaced abscissas are easy to compute.

A computational weakness of Romberg integration is that twice as many function evaluations are needed to decrease the error from $\boldsymbol{O}(h^{2N})$ to $\boldsymbol{O}(h^{2N+2})$. The use of the sequential rules will help keep the number of computations down. The development of Romberg integration relies on the theoretical assumption that, if $f \in C^N[a, b]$ for all N, then the error term for the trapezoidal rule can be represented in a series involving only even powers of h; that is,

$$\int_a^b f(x)\,dx = T(f, h) + E_T(f, h), \tag{18}$$

where

$$E_T(f, h) = a_1 h^2 + a_2 h^4 + a_3 h^6 + \cdots . \tag{19}$$

Since only even powers of h can occur in (19), the Richardson improvement process is used successively first to eliminate a_1, next to eliminate a_2, then to eliminate a_3, and so on. This process generates quadrature formulas whose error terms have even orders $\boldsymbol{O}(h^4)$, $\boldsymbol{O}(h^6)$, $\boldsymbol{O}(h^8)$, and so on. We shall show that the first improvement is Simpson's rule for $2M$ intervals. Start with $T(f, 2h)$ and $T(f, h)$ and the equations

$$\int_a^b f(x)\,dx = T(f, 2h) + a_1 4h^2 + a_2 16h^4 + a_3 64h^6 + \cdots \tag{20}$$

and

$$\int_a^b f(x)\,dx = T(f,h) + a_1h^2 + a_2h^4 + a_3h^6 + \cdots. \tag{21}$$

Multiply equation (21) by 4 and obtain

$$4\int_a^b f(x)\,dx = 4T(f,h) + a_1 4h^2 + a_2 4h^4 + a_3 4h^6 + \cdots. \tag{22}$$

Eliminate a_1 by subtracting (20) from (22). The result is

$$3\int_a^b f(x)\,dx = 4T(f,h) - T(f,2h) - a_2 12h^4 - a_3 60h^6 - \cdots. \tag{23}$$

Now divide equation (23) by 3 and rename the coefficients in the series:

$$\int_a^b f(x)\,dx = \frac{4T(f,h) - T(f,2h)}{3} + b_1h^4 + b_2h^6 + \cdots. \tag{24}$$

As noted in (6), the first quantity on the right side of (24) is Simpson's rule $S(f,h)$. This shows that $E_S(f,h)$ involves only even powers of h:

$$\int_a^b f(x)\,dx = S(f,h) + b_1h^4 + b_2h^6 + b_3h^8 + \cdots. \tag{25}$$

To show that the second improvement is Boole's rule, start with (25) and write down the formula involving $S(f,2h)$:

$$\int_a^b f(x)\,dx = S(f,2h) + b_1 16h^4 + b_2 64h^6 + b_3 256h^8 + \cdots. \tag{26}$$

When b_1 is eliminated from (25) and (26), the result involves Boole's rule:

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{16S(f,h) - S(f,2h)}{15} - \frac{b_2 48h^6}{15} - \frac{b_3 240h^8}{15} - \cdots \\ &= B(f,h) - \frac{b_2 48h^6}{15} - \frac{b_3 240h^8}{15} - \cdots.\end{aligned} \tag{27}$$

The general pattern for Romberg integration relies on Lemma 7.1.

Lemma 7.1 (Richardson's Improvement for Romberg Integration). Given two approximations $R(2h, K-1)$ and $R(h, K-1)$ for the quantity Q that satisfy

$$Q = R(h, K-1) + c_1 h^{2K} + c_2 h^{2K+2} + \cdots \tag{28}$$

and

$$Q = R(2h, K-1) + c_1 4^K h^{2K} + c_2 4^{K+1} h^{2K+2} + \cdots, \tag{29}$$

an improved approximation has the form

$$Q = \frac{4^K R(h, K-1) - R(2h, K-1)}{4^K - 1} + \boldsymbol{O}(h^{2K+2}). \tag{30}$$

Proof. The proof is straightforward and is left for the reader. •

Definition 7.4. Define the sequence $\{R(J, K) : J \geq K\}_{J=0}^{\infty}$ of quadrature formulas for $f(x)$ over $[a, b]$ as follows

$$\begin{aligned} R(J, 0) &= T(J) && \text{for } J \geq 0, \text{ is the sequential trapezoidal rule.} \\ R(J, 1) &= S(J) && \text{for } J \geq 1, \text{ is the sequential Simpson rule.} \\ R(J, 2) &= B(J) && \text{for } J \geq 2, \text{ is the sequential Boole's rule.} \end{aligned} \tag{31}$$ ▲

The starting rules, $\{R(J, 0)\}$, are used to generate the first improvement, $\{R(J, 1)\}$, which in turn is used to generate the second improvement, $\{R(J, 2)\}$. We have already seen the patterns

$$\begin{aligned} R(J, 1) &= \frac{4^1 R(J, 0) - R(J-1, 0)}{4^1 - 1} && \text{for } J \geq 1 \\ R(J, 2) &= \frac{4^2 R(J, 1) - R(J-1, 1)}{4^2 - 1} && \text{for } J \geq 2, \end{aligned} \tag{32}$$

which are the rules in (24) and (27) stated using the notation in (31). The general rule for constructing improvements is

$$R(J, K) = \frac{4^K R(J, K-1) - R(J-1, K-1)}{4^K - 1} \quad \text{for } J \geq K. \tag{33}$$

Table 7.5 Romberg Integration Tableau

J	$R(J,0)$ Trapezoidal rule	$R(J,1)$ Simpson's rule	$R(J,2)$ Boole's rule	$R(J,3)$ Third improvement	$R(J,4)$ Fourth improvement
0	$R(0,0)$				
1	$R(1,0)$	$R(1,1)$			
2	$R(2,0)$	$R(2,1)$	$R(2,2)$		
3	$R(3,0)$	$R(3,1)$	$R(3,2)$	$R(3,3)$	
4	$R(4,0)$	$R(4,1)$	$R(4,2)$	$R(4,3)$	$R(4,4)$

Table 7.6 Romberg Integration Tableau for Example 7.14

J	$R(J,0)$ Trapezoidal rule	$R(J,1)$ Simpson's rule	$R(J,2)$ Boole's rule	$R(J,3)$ Third improvement
0	0.785398163397			
1	1.726812656758	2.040617487878		
2	1.960534166564	2.038441336499	2.038296259740	
3	2.018793948078	2.038213875249	2.038198711166	2.038197162776
4	2.033347341805	2.038198473047	2.038197446234	2.038197426156
5	2.036984954990	2.038197492719	2.038197427363	2.038197427064

For computational purposes, the values $R(J, K)$ are arranged in the Romberg integration tableau given in Table 7.5.

Example 7.14. Use Romberg integration to find approximations for the definite integral

$$\int_0^{\pi/2} (x^2 + x + 1)\cos(x)\,dx = -2 + \frac{\pi}{2} + \frac{\pi^2}{4} = 2.038197427067\ldots.$$

The computations are given in Table 7.6. In each column the numbers are converging to the value $2.038197427067\ldots$. The values in the Simpson's rule column converge faster than the values in the trapezoidal rule column. For this example, convergence in columns to the right is faster than the adjacent column to the left.

Convergence of the Romberg values in Table 7.6 is easier to see if we look at the error terms $E(J, K) = -2+\pi/2+\pi^2/4-R(J, K)$. Suppose that the interval width is $h = b-a$ and that the higher derivatives of $f(x)$ are of the same magnitude. The error in column K of the Romberg table diminishes by about a factor of $1/2^{2K+2} = 1/4^{K+1}$ as one progresses down its rows. The errors $E(J, 0)$ diminish by a factor of $1/4$, the errors $E(J, 1)$ diminish by a factor of $1/16$, and so on. This can be observed by inspecting the entries $\{E(J, K)\}$ in Table 7.7. ∎

Table 7.7 Romberg Error Tableau for Example 7.14

J	h	$E(J,0) = \boldsymbol{O}(h^2)$	$E(J,1) = \boldsymbol{O}(h^4)$	$E(J,2) = \boldsymbol{O}(h^6)$	$E(J,3) = \boldsymbol{O}(h^8)$
0	$b-a$	−1.252799263670			
1	$\frac{b-a}{2}$	−0.311384770309	0.002420060811		
2	$\frac{b-a}{4}$	−0.077663260503	0.000243909432	0.000098832673	
3	$\frac{b-a}{8}$	−0.019403478989	0.000016448182	0.000001284099	−0.000000264291
4	$\frac{b-a}{16}$	−0.004850085262	0.000001045980	0.000000019167	−0.000000000912
5	$\frac{b-a}{32}$	−0.001212472077	0.000000065651	0.000000000296	−0.000000000003

Theorem 7.7 (Precision of Romberg Integration). Assume that $f \in C^{2K+2}[a,b]$. Then the truncation error term for the Romberg approximation is given in the formula

$$\int_a^b f(x)\,dx = R(J,K) + b_K h^{2K+2} f^{(2K+2)}(c_{J,K}) \tag{34}$$
$$= R(J,K) + \boldsymbol{O}(h^{2K+2}),$$

where $h = (b-a)/2^J$, b_K is a constant that depends on K, and $c_{J,K} \in [a,b]$.

Example 7.15. Apply Theorem 7.7 and show that

$$\int_0^2 10x^9\,dx = 1024 \equiv R(4,4).$$

The integrand is $f(x) = 10x^9$, and $f^{(10)}(x) \equiv 0$. Thus the value $K = 4$ will make the error term identically zero. A numerical computation will produce $R(4,4) = 1024$. ∎

Numerical Methods Using Matlab, 4th Edition, 2004

John H. Mathews and Kurtis K. Fink

ISBN: 0-13-065248-2

Prentice-Hall Inc.
Upper Saddle River, New Jersey, USA
http://vig.prenhall.com/