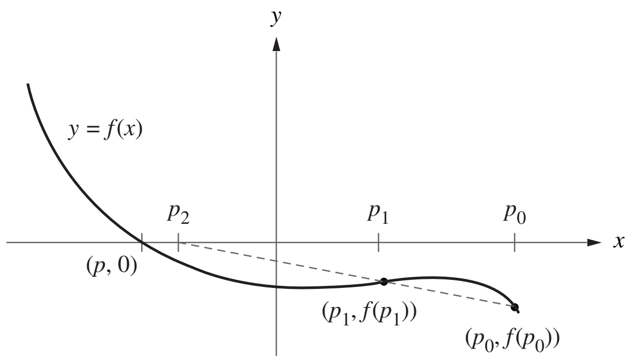


**Figure 2.16** The geometric construction of $p_2$ for the secant method.

## Secant Method

The Newton-Raphson algorithm requires the evaluation of two functions per iteration, $f(p_{k-1})$ and $f'(p_{k-1})$. Traditionally, the calculation of derivatives of elementary functions could involve considerable effort. But with modern computer algebra software packages, this has become less of an issue. Still many functions have nonelementary forms (integrals, sums, etc.), and it is desirable to have a method that converges almost as fast as Newton's method yet involves only evaluations of $f(x)$ and not of $f'(x)$. The secant method will require only one evaluation of $f(x)$ per step and at a simple root has an order of convergence $R \approx 1.618033989$. It is almost as fast as Newton's method, which has order 2.

The formula involved in the secant method is the same one that was used in the regula falsi method, except that the logical decisions regarding how to define each succeeding term are different. Two initial points $(p_0, f(p_0))$ and $(p_1, f(p_1))$ near the point $(p, 0)$ are needed, as shown in Figure 2.16. Define $p_2$ to be the abscissa

**Table 2.7** Convergence of the Secant Method at a Simple Root

| $k$ | $p_k$ | $p_{k+1} - p_k$ | $E_k = p - p_k$ | $\dfrac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^{1.618}}$ |
|---|---|---|---|---|
| 0 | −2.600000000 | 0.200000000 | 0.600000000 | 0.914152831 |
| 1 | −2.400000000 | 0.293401015 | 0.400000000 | 0.469497765 |
| 2 | −2.106598985 | 0.083957573 | 0.106598985 | 0.847290012 |
| 3 | −2.022641412 | 0.021130314 | 0.022641412 | 0.693608922 |
| 4 | −2.001511098 | 0.001488561 | 0.001511098 | 0.825841116 |
| 5 | −2.000022537 | 0.000022515 | 0.000022537 | 0.727100987 |
| 6 | −2.000000022 | 0.000000022 | 0.000000022 | |
| 7 | −2.000000000 | 0.000000000 | 0.000000000 | |

of the point of intersection of the line through these two points and the $x$-axis; then Figure 2.16 shows that $p_2$ will be closer to $p$ than to either $p_0$ or $p_1$. The equation relating $p_2$, $p_1$, and $p_0$ is found by considering the slope

(24) $$m = \frac{f(p_1) - f(p_0)}{p_1 - p_0} \quad \text{and} \quad m = \frac{0 - f(p_1)}{p_2 - p_1}.$$

The values of $m$ in (25) are the slope of the secant line through the first two approximations and the slope of the line through $(p_1, f(p_1))$ and $(p_2, 0)$, respectively. Set the right-hand sides equal in (25) and solve for $p_2 = g(p_1, p_0)$ and get

(25) $$p_2 = g(p_1, p_0) = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}.$$

The general term is given by the two-point iteration formula

(26) $$p_{k+1} = g(p_k, p_{k-1}) = p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})}.$$

**Example 2.16 (Secant Method at a Simple Root).** Start with $p_0 = -2.6$ and $p_1 = -2.4$ and use the secant method to find the root $p = -2$ of the polynomial function $f(x) = x^3 - 3x + 2$.

In this case the iteration formula (27) is

(27) $$p_{k+1} = g(p_k, p_{k-1}) = p_k - \frac{(p_k^3 - 3p_k + 2)(p_k - p_{k-1})}{p_k^3 - p_{k-1}^3 - 3p_k + 3p_{k-1}}.$$

This can be algebraically manipulated to obtain

(28) $$p_{k+1} = g(p_k, p_{k-1}) = \frac{p_k^2 p_{k-1} + p_k p_{k-1}^2 - 2}{p_k^2 + p_k p_{k-1} + p_{k-1}^2 - 3}.$$

The sequence of iterates is given in Table 2.7. ∎

There is a relationship between the secant method and Newton's method. For a polynomial function $f(x)$, the secant method two-point formula $p_{k+1} = g(p_k, p_{k-1})$ will reduce to Newton's one-point formula $p_{k+1} = g(p_k)$ if $p_k$ is replaced by $p_{k-1}$. Indeed, if we replace $p_k$ by $p_{k-1}$ in (29), then the right side becomes the same as the right side of (22) in Example 2.14.

Proofs about the rate of convergence of the secant method can be found in advanced texts on numerical analysis. Let us state that the error terms satisfy the relationship

$$|E_{k+1}| \approx |E_k|^{1.618} \left|\frac{f''(p)}{2f'(p)}\right|^{0.618} \tag{29}$$

where the order of convergence is $R = (1+\sqrt{5})/2 \approx 1.618$ and the relation in (30) is valid only at simple roots.

To check this, we make use of Example 2.16 and the specific values

$$|p - p_5| = 0.000022537$$
$$|p - p_4|^{1.618} = 0.001511098^{1.618} = 0.000027296,$$

and

$$A = |f''(-2)/2f'(-2)|^{0.618} = (2/3)^{0.618} = 0.778351205.$$

Combine these and it is easy to see that

$$|p - p_5| = 0.000022537 \approx 0.000021246 = A|p - p_4|^{1.618}.$$

## Accelerated Convergence

We could hope that there are root-finding techniques that converge faster than linearly when $p$ is a root of order $M$. Our final result shows that a modification can be made to Newton's method so that convergence becomes quadratic at a multiple root.

**Theorem 2.7 (Acceleration of Newton-Raphson Iteration).** Suppose that the Newton-Raphson algorithm produces a sequence that converges linearly to the root $x = p$ of order $M > 1$. Then the Newton-Raphson iteration formula

$$p_k = p_{k-1} - \frac{Mf(p_{k-1})}{f'(p_{k-1})} \tag{30}$$

will produce a sequence $\{p_k\}_{k=0}^{\infty}$ that converges quadratically to $p$.

**Table 2.8** Acceleration of Convergence at a Double Root

| $k$ | $p_k$ | $p_{k+1} - p_k$ | $E_k = p - p_k$ | $\dfrac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^2}$ |
|---|---|---|---|---|
| 0 | 1.200000000 | −0.193939394 | −0.200000000 | 0.151515150 |
| 1 | 1.006060606 | −0.006054519 | −0.006060606 | 0.165718578 |
| 2 | 1.000006087 | −0.000006087 | −0.000006087 | |
| 3 | 1.000000000 | 0.000000000 | 0.000000000 | |

**Table 2.9** Comparison of the Speed of Convergence

| Method | Special considerations | Relation between successive error terms |
|---|---|---|
| Bisection | | $E_{k+1} \approx \frac{1}{2}\lvert E_k\rvert$ |
| Regula falsi | | $E_{k+1} \approx A\lvert E_k\rvert$ |
| Secant method | Multiple root | $E_{k+1} \approx A\lvert E_k\rvert$ |
| Newton-Raphson | Multiple root | $E_{k+1} \approx A\lvert E_k\rvert$ |
| Secant method | Simple root | $E_{k+1} \approx A\lvert E_k\rvert^{1.618}$ |
| Newton-Raphson | Simple root | $E_{k+1} \approx A\lvert E_k\rvert^2$ |
| Accelerated Newton-Raphson | Multiple root | $E_{k+1} \approx A\lvert E_k\rvert^2$ |

**Example 2.17 (Acceleration of Convergence at a Double Root).** Start with $p_0 = 1.2$ and use accelerated Newton-Raphson iteration to find the double root $p = 1$ of $f(x) = x^3 - 3x + 2$.

Since $M = 2$, the acceleration formula (31) becomes

$$p_k = p_{k-1} - 2\frac{f(p_{k-1})}{f'(p_{k-1})} = \frac{p_{k-1}^3 + 3p_{k-1} - 4}{3p_{k-1}^2 - 3},$$

and we obtain the values in Table 2.8. ■

Table 2.9 compares the speed of convergence of the various root-finding methods that we have studied so far. The value of the constant $A$ is different for each method.

**Numerical Methods Using Matlab, 4th Edition, 2004**

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ISBN: 0-13-065248-2

Prentice-Hall Inc.
Upper Saddle River, New Jersey, USA
http://vig.prenhall.com/