9.4 Taylor Series Method

The Taylor series method is of general applicability, and it is the standard to which we compare the accuracy of the various other numerical methods for solving an I.V.P. It can be devised to have any specified degree of accuracy. We start by reformulating Taylor's theorem in a form that is suitable for solving differential equations.

Theorem 9.5 (Taylor's Theorem). Assume that $y(t) \in C^{N+1}[t_0, b]$ and that $y(t)$ has a Taylor series expansion of order N about the fixed value $t = t_k \in [t_0, b]$:

$$y(t_k + h) = y(t_k) + hT_N(t_k, y(t_k)) + \boldsymbol{O}(h^{N+1}), \tag{1}$$

where

$$T_N(t_k, y(t_k)) = \sum_{j=1}^{N} \frac{y^{(j)}(t_k)}{j!} h^{j-1} \tag{2}$$

and $y^{(j)}(t) = f^{(j-1)}(t, y(t))$ denotes the $(j-1)$st total derivative of the function f with respect to t. The formulas for the derivatives can be computed recursively:

$$\begin{aligned}
y'(t) &= f \\
y''(t) &= f_t + f_y y' = f_t + f_y f \\
y^{(3)}(t) &= f_{tt} + 2f_{ty} y' + f_y y'' + f_{yy}(y')^2 \\
&= f_{tt} + 2f_{ty} f + f_{yy} f^2 + f_y(f_t + f_y f) \\
y^{(4)}(t) &= f_{ttt} + 3f_{tty} y' + 3f_{tyy}(y')^2 + 3f_{ty} y'' \\
&\quad + f_y y''' + 3f_{yy} y' y'' + f_{yyy}(y')^3 \\
&= (f_{ttt} + 3f_{tty} f + 3f_{tyy} f^2 + f_{yyy} f^3) + f_y(f_{tt} + 2f_{ty} f + f_{yy} f^2) \\
&\quad + 3(f_t + f_y f)(f_{ty} + f_{yy} f) + f_y^2(f_t + f_y f)
\end{aligned} \tag{3}$$

and, in general,

$$y^{(N)}(t) = P^{(N-1)} f(t, y(t)), \tag{4}$$

where P is the derivative operator

$$P = \left(\frac{\partial}{\partial t} + f \frac{\partial}{\partial y} \right).$$

The approximate numerical solution to the I.V.P. $y'(t) = f(t, y)$ over $[t_0, t_M]$ is derived by using formula (1) on each subinterval $[t_k, t_{k+1}]$. The general step for Taylor's method of order N is

$$y_{k+1} = y_k + d_1 h + \frac{d_2 h^2}{2!} + \frac{d_3 h^3}{3!} + \cdots + \frac{d_N h^N}{N!}, \tag{5}$$

where $d_j = y^{(j)}(t_k)$ for $j = 1, 2, \ldots, N$ at each step $k = 0, 1, \ldots, M - 1$.

The Taylor method of order N has the property that the final global error (F.G.E.) is of the order $\boldsymbol{O}(h^{N+1})$; hence N can be chosen as large as necessary to make this error as small as desired. If the order N is fixed, it is theoretically possible to a priori determine the step size h so that the F.G.E. will be as small as desired. However, in practice we usually compute two sets of approximations using step sizes h and $h/2$ and compare the results.

Theorem 9.6 (Precision of Taylor's Method of Order N). Assume that $y(t)$ is the solution to the I.V.P.. If $y(t) \in C^{N+1}[t_0, b]$ and $\{(t_k, y_k)\}_{k=0}^{M}$ is the sequence of approximations generated by Taylor's method of order N, then

$$\begin{aligned} |e_k| &= |y(t_k) - y_k| = \boldsymbol{O}(h^N), \\ |\epsilon_{k+1}| &= |y(t_{k+1}) - y_k - hT_N(t_k, y_k)| = \boldsymbol{O}(h^{N+1}). \end{aligned} \tag{6}$$

In particular, the F.G.E. at the end of the interval will satisfy

$$E(y(b), h) = |y(b) - y_M| = \boldsymbol{O}(h^N). \tag{7}$$

Examples 9.8 and 9.9 illustrate Theorem 9.6 for the case $N = 4$. If approximations are computed using the step sizes h and $h/2$, we should have

$$E(y(b), h) \approx Ch^4 \tag{8}$$

for the larger step size, and

$$E\left(y(b), \frac{h}{2}\right) \approx C\frac{h^4}{16} = \frac{1}{16}Ch^4 \approx \frac{1}{16}E\left(y\left(b\right), h\right). \tag{9}$$

Hence the idea in Theorem 9.6 is that if the step size in the Taylor method of order 4 is reduced by a factor of $\frac{1}{2}$, the overall F.G.E. will be reduced by about $\frac{1}{16}$.

Example 9.8. Use the Taylor method of order $N = 4$ to solve $y' = (t - y)/2$ on $[0, 3]$ with $y(0) = 1$. Compare solutions for $h = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$.

The derivatives of $y(t)$ must first be determined. Recall that the solution $y(t)$ is a function of t, and differentiate the formula $y'(t) = f(t, y(t))$ with respect to t to get

$y^{(2)}(t)$. Then continue the process to obtain the higher derivatives.

$$\begin{aligned}
y'(t) &= \frac{t-y}{2},\\
y^{(2)}(t) &= \frac{d}{dt}\left(\frac{t-y}{2}\right) = \frac{1-y'}{2} = \frac{1-(t-y)/2}{2} = \frac{2-t+y}{4},\\
y^{(3)}(t) &= \frac{d}{dt}\left(\frac{2-t+y}{4}\right) = \frac{0-1+y'}{4} = \frac{-1+(t-y)/2}{4} = \frac{-2+t-y}{8},\\
y^{(4)}(t) &= \frac{d}{dt}\left(\frac{-2+t-y}{8}\right) = \frac{-0+1-y'}{8} = \frac{1-(t-y)/2}{8} = \frac{2-t+y}{16}.
\end{aligned}$$

To find y_1, the derivatives given above must be evaluated at the point $(t_0, y_0) = (0, 1)$. Calculation reveals that

$$\begin{aligned}
d_1 &= y'(0) = \frac{0.0-1.0}{2} = -0.5,\\
d_2 &= y^{(2)}(0) = \frac{2.0-0.0+1.0}{4} = 0.75,\\
d_3 &= y^{(3)}(0) = \frac{-2.0+0.0-1.0}{8} = -0.375,\\
d_4 &= y^{(4)}(0) = \frac{2.0-0.0+1.0}{16} = 0.1875.
\end{aligned}$$

Next the derivatives $\{d_j\}$ are substituted into (5) with $h = 0.25$, and nested multiplication is used to compute the value y_1:

$$\begin{aligned}
y_1 &= 1.0 + 0.25\left(-0.5 + 0.25\left(\frac{0.75}{2} + 0.25\left(\frac{-0.375}{6} + 0.25\left(\frac{0.1875}{24}\right)\right)\right)\right)\\
&= 0.8974915.
\end{aligned}$$

The computed solution point is $(t_1, y_1) = (0.25, 0.8974915)$.

To determine y_2, the derivatives $\{d_j\}$ must now be evaluated at the point $(t_1, y_1) = (0.25, 0.8974915)$. The calculations are starting to require a considerable amount of computational effort and are tedious to do by hand. Calculation reveals that

$$\begin{aligned}
d_1 &= y'(0.25) = \frac{0.25-0.8974915}{2} = -0.3237458,\\
d_2 &= y^{(2)}(0.25) = \frac{2.0-0.25+0.8974915}{4} = 0.6618729,\\
d_3 &= y^{(3)}(0.25) = \frac{-2.0+0.25-0.8974915}{8} = -0.3309364,\\
d_4 &= y^{(4)}(0.25) = \frac{2.0-0.25+0.8974915}{16} = 0.1654682.
\end{aligned}$$

Now these derivatives $\{d_j\}$ are substituted into (5) with $h = 0.25$, and nested multiplication

Table 9.6 Comparison of the Taylor Solutions of Order $N = 4$ for $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

	y_k				
t_k	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$y(t_k)$ Exact
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9432392	0.9432392
0.25			0.8974915	0.8974908	0.8974917
0.375				0.8620874	0.8620874
0.50		0.8364258	0.8364037	0.8364024	0.8364023
0.75			0.8118696	0.8118679	0.8118678
1.00	0.8203125	0.8196285	0.8195940	0.8195921	0.8195920
1.50		0.9171423	0.9171021	0.9170998	0.9170997
2.00	1.1045125	1.1036826	1.1036408	1.1036385	1.1036383
2.50		1.3595575	1.3595168	1.3595145	1.3595144
3.00	1.6701860	1.6694308	1.6693928	1.6693906	1.6693905

is used to compute the value y_2:

$$\begin{aligned} y_2 &= 0.8974915 + 0.25\Bigg(-0.3237458 \\ &\qquad + 0.25\left(\frac{0.6618729}{2} + 0.25\left(\frac{-0.3309364}{6} + 0.25\left(\frac{0.1654682}{24}\right)\right)\right)\Bigg) \\ &= 0.8364037. \end{aligned}$$

The solution point is $(t_2, y_2) = (0.50, 0.8364037)$. Table 9.6 gives solution values at selected abscissas using various step sizes. ■

Example 9.9. Compare the F.G.E. for the Taylor solutions to $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$ given in Example 9.8.

Table 9.7 gives the F.G.E. for these step sizes and shows that the error in the approximation $y(3)$ decreases by about $\frac{1}{16}$ when the step size is reduced by a factor of $\frac{1}{2}$:

$$E(y(3), h) = y(3) - y_M = \boldsymbol{O}(h^4) \approx Ch^4, \qquad \text{where } C = -0.000614. \quad ■$$

The following program requires that the derivatives y', y'', y''', and y'''' be saved in an M-file named `df`. For example, the following M-file would save the derivatives from Example 9.8 in the format required by Program 9.3.

```
function z=df(t,y)
z=[(t-y)/2 (2-t+y)/4 (-2+t-y)/8 (2-t+y)/16];
```

Table 9.7 Relation between Step Size and F.G.E. for the Taylor Solutions to $y' = (t - y)/2$ over $[0, 3]$

Step size, h	Number of steps, M	Approximation to $y(3)$, y_M	F.G.E. Error at $t = 3$, $y(3) - y_M$	$\boldsymbol{O}(h^2) \approx Ch^4$ where $C = -0.000614$
1	3	1.6701860	−0.0007955	−0.0006140
$\frac{1}{2}$	6	1.6694308	−0.0000403	−0.0000384
$\frac{1}{4}$	12	1.6693928	−0.0000023	−0.0000024
$\frac{1}{8}$	24	1.6693906	−0.0000001	−0.0000001

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