Statement of Research Interests

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Associated prime ideals have played an important role in commutative ring theory ever since their introduction. A noncommutative theory of associated primes has more recently been developed, and my research in part concerns properties of associated primes over noncommutative rings.

Given a module M over a commutative ring R, its associated primes are defined simply to be the prime ideals which are the annihilator of an element $m \in M$. As a standard tool in commutative algebra, associated primes now boast a rich and extensive theory and range of applications. Most notably, they appear center stage in the well-known theory of primary decomposition, which features the decomposition of Noetherian modules as an intersection of "nicer" modules known as primary modules, each of which has a single prime ideal *associated* to it. Primary decomposition has far-reaching applications, not only in algebra, but also in geometry. Specifically, in the case of a polynomial ring in r commuting variables, the primary decomposition of an ideal (a purely algebra-theoretic concept) can be translated into the language of geometry via a correspondence which associates to the primary decomposition a union of irreducible varieties in affine r-space.

The issue of ascertaining how various ring-theoretic and module-theoretic concepts, such as the associated primes, behave under various types of change of rings, such as subrings, polynomial rings, matrix rings, and localizations, has always been of fundamental interest among ring theorists. A fine illustration of this can be found in ([L], Section 6F). Much of my doctoral dissertation is concerned with how the associated primes are affected by these types of change of rings, with particular focus on the noncommutative aspects of this theory.

Carl Faith proved in [F] that if R is commutative, then the associated primes of $R[x]$ (viewed as a module over itself) are precisely the ideals of the form $\mathfrak{p}[x]$, where $\mathfrak{p}$ is an associated prime of R (viewed as a module over itself). In [A1] and [A2] I generalized this result in several ways, including a consideration of general modules, rings R which are noncommutative, and polynomial extensions which are twisted in some way so that the variable x may not commute with the coefficients in the base ring R. The latter case includes skew polynomial rings and differential polynomial rings.

There is an obvious obstruction in trying to apply the standard commutative definition of an associated prime in the noncommutative setting. In general, for an element m in a right R-module M, its annihilator is only a right ideal. Since prime ideals[1] must be two-sided, this raises the issue of how to properly extend the notion of associated primes to noncommutative ring theory. The definitions below provide the answer.

Definition. A right module $N \neq 0$ over a (possibly) noncommutative ring R is *prime* if the annihilator of N is the same as the annihilator of N' for every nonzero (right) submodule $N' \subseteq N$.

[1] Recall that a *prime ideal* $\mathfrak{p}$ in a noncommutative ring R is a proper, two-sided ideal such that for $a, b \in R$, $aRb \subseteq \mathfrak{p}$ implies that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

It is easy to see that the annihilator of a prime module is a prime ideal.

Definition. We say $\mathfrak{p}$ is an *associated prime* of a right module M if there exists a prime submodule $N \subseteq M$ whose annihilator is $\mathfrak{p}$.

If R is commutative, this definition agrees with the usual one. An excellent introduction to associated primes over noncommutative rings can be found in ([L], Section 3F) which includes an important application to the study of isomorphism classes of indecomposable injective modules over some noncommutative rings. In studying the dual theory of so-called attached primes, I was able to give a similar application (discussed below).

As mentioned above, my work on associated primes under polynomial extensions includes a study of extensions of skew type, differential type, or both. To summarize, recall that if $\sigma : R \to R$ is an endomorphism and $\delta : R \to R$ is an additive map satisfying $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$, then δ is called a *σ-derivation.* The Ore extension ring $R[x; \sigma, \delta]$ consists of all polynomials $\sum_{i=0}^{n} a_i x^i$ $(a_i \in R)$, which are multiplied using the rule $xa = \sigma(a)x + \delta(a)$ for each $a \in R$.

There has been tremendous interest in the study of Ore extension rings, especially as rings of differential operators. These rings also arise as universal enveloping algebras of Lie algebras, and as pseudo-linear ring extensions[2]. (Cyclic algebras, for example, arise in exactly this way).

One of the main difficulties in the study of Ore extension rings is the issue of determining the ideals and the prime ideals in such rings. This has remained a big challenge which various authors have examined (for example, see [G-L]). To give an indication of the problems that arise, suppose that $\mathfrak{p}$ is a prime ideal of R. Then we observe that $\mathfrak{p}[x]$ need not even be an ideal of the Ore extension ring in general, since $\mathfrak{p}$ need not be invariant under σ and δ: if $a \in \mathfrak{p}$, then $x \cdot a = \sigma(a)x + \delta(a)$, and the coefficients $\sigma(a)$ and $\delta(a)$ need not lie in $\mathfrak{p}$.

In view of the above difficulty, we necessarily must place some constraints on the σ and δ we consider. We will actually phrase these as constraints on the given right module M.

Definition. Given M, σ, and δ as above, we say M is *σ-compatible* if for each $m \in M, r \in R$, we have $mr = 0 \iff m\sigma(r) = 0$. Similarly, we say that M is *δ-compatible* if for each $m \in M, r \in R$, we have $mr = 0 \Longrightarrow m\delta(r) = 0$. If M is both σ and δ-compatible, we say M is *(σ, δ)-compatible.*

Note that if M is a right R-module, we can make $M[x]$ into a right $R[x; \sigma, \delta]$-module by extending the action of R on M in the obvious way, applying the twist law wherever appropriate. One of my main results on associated primes is the following.

Theorem. If M is (σ, δ)-compatible, then the associated primes of $M[x]$ (viewed as a module over the Ore extension) are precisely the ideals $\mathfrak{p}[x]$, where $\mathfrak{p}$ is an associated prime of M.

[2] Given a division ring K, an overring $L \supseteq K$ is called a *(left) pseudo-linear ring extension* if there exists $s \in L$ such that there exists $n \geq 2$ such that $\{1, s, s^2, ..., s^{n-1}\}$ is a left K-basis of L and $s \cdot K \subseteq K \cdot s + K$.

This result is proved in [A2], and examples are given to illustrate the necessity of the compatibility assumptions on M. Of course, the compatibility is automatic if $\sigma = Id$ and $\delta = 0$, and the theorem is already of interest in this untwisted case. The theorem also prompts many related questions, some of which are addressed in [A1] and [A2], and some of which remain open problems which could potentially form the basis for some of my future research projects. These include questions on how the associated primes behave under power series extensions, Laurent series extensions, and inverse polynomial extensions.

In addition to the study of associated primes in noncommutative ring theory, I have also been interested in a related notion, the *attached primes*, which are at the core of a dual theory pioneered by I.G. Macdonald in his landmark 1973 paper [M]. The basic idea is to replace primary decomposition with a different sort of decomposition which is in some sense dual. Namely, starting with a module M over a commutative ring R, one can try to decompose M as a sum of "nicer" submodules (which play a role dual to the primary modules): $M = M_1 + M_2 + ... + M_n$. Such a decomposition is referred to as a *secondary representation.* We can "attach" a single prime ideal $\mathfrak{p}_i$ to each M_i and thus obtain the set of "attached" primes of M. We do not have space here to cover this theory, but an excellent exposition is available in [M], and an extensive literature on this subject has been generated. Many, but not all, of the results on associated primes have natural dual statements for the attached primes.

When I learned this theory, I was immediately curious about possible extensions of it to noncommutative rings, in the hopes that some of the noncommutative results on associated primes might also have natural duals. I made the following definitions in [A3], motivated by the first two definitions above.

Definition. A right module $N \neq 0$ over a (possibly) noncommutative ring R is *coprime* if the annihilator of N is the same as the annihilator of Q for every nonzero (right) quotient Q of N.

Again, the annihilator of such a module is a prime ideal.

Definition. We say $\mathfrak{p}$ is an *attached prime* of a right module M if there exists a coprime quotient of M whose annihilator is $\mathfrak{p}$.

I showed in [A3] that this definition agrees with Macdonald's whenever the latter applies and proved a number of basic results on this new, more general version of the attached primes which provide further testimony to their dual nature.

Given the discussion above on the behavior of associated primes under polynomial extensions, we are prompted to ask: do the attached primes exhibit any predictable behavior under polynomial extensions? For the classical situation of Macdonald, the first result of this nature was proved by L. Melkersson in [Me]. His result is dual to Faith's and uses *inverse* polynomials! Given a module M over a commutative ring R, one can make the set of inverse polynomials with coefficients in M, $M[x^{-1}]$, into an $R[x]$ module by declaring that for $i, j \geq 0$, $(mx^{-j})(rx^i) = mrx^{i-j}$ if $i - j \leq 0$ and 0 otherwise. Melkersson proved that the attached primes of $M[x^{-1}]$ (viewed as an $R[x]$-module) are precisely all ideals of the form $\mathfrak{p}[x]$, where $\mathfrak{p}$ is an attached prime of M.

I showed in Example 3.5 of [A3], however, that Melkersson's result does not hold in the noncommutative setting. One must additionally assume that the ring R is right perfect, a class of rings which includes, for example, all Artinian rings. With this assumption, one can achieve the desired generalization and even allow for the possible presence of a "twist" automorphism σ. For simplicity, the result is stated here in untwisted form.

Theorem. Let M be a right module over any ring R. Then the attached primes of $M[x^{-1}]$ (viewed as a module over $R[x]$) are precisely the ideals $\mathfrak{p}[x]$, where $\mathfrak{p}$ is an attached prime of M.

The entire theory of attached primes becomes much more tractable over right perfect rings ([A3], Section 4). In this setting, I used the attached primes to induce a bijection between isomorphism classes of *hollow projective* modules and the maximal ideals of the ring, just as the associated primes have been used to relate isomorphism classes of indecomposable injectives to prime ideals for some classes of noncommutative rings.

There are many interesting conjectures about how the commutative results on associated and attached primes might extend to the noncommutative theories I have described here. I expect my future research to include various investigations of this type. For example, if R is a commutative Noetherian ring and I is an ideal, the sequences $A(n) := \mathrm{Ass}(R/I^n)$ and $B(n) := \mathrm{Ass}(I^{n-1}/I^n)$ have been extensively studied (as have dual sequences regarding the attached primes). M. Brodmann showed in [B] that these sequences stabilize, and S. McAdam and P. Eakin obtained numerous results about the eventual constant values of these sequences in [M-E]. One can naturally ask whether these results have noncommutative analogs. To answer this, and to stock up our supply of examples, it would be helpful to develop the notions we have discussed from a more computational point of view, and I would like to pursue this. Finally, recall that in the commutative case, associated primes are tied closely to primary decomposition and algebraic geometry. It would be of interest to investigate possible parallel connections between noncommutative algebra and noncommutative geometry, where the associated primes would be expected to play an especially visible role.

References

[A1] S. Annin *Associated primes over skew polynomial rings*, Communications in Algebra, **30** (2002), p. 2511-2528.

[A2] S. Annin *Associated primes over Ore polynomial rings*, Journal of Algebra and its Applications, **3** (2004, no. 2, p. 193-205.

[A3] S. Annin *Attached primes over noncommutative rings*, preprint.

[B] M. Brodmann *Asymptotic stability of Ass*(R/I^n), Proc. Amer. Math. Soc., 74 (1), p. 16-18, c. 1979.

[F] C. Faith *Associated primes in commutative polynomial rings*, Communications in Algebra, 28 (8), p. 3983-3986, c. 2000.

[G-L] K. Goodearl and E.S. Letzter *Prime ideals in skew and q-skew polynomial rings* Mem. Amer. Math. Soc. 109(521), vi + 106pp., c. 1994.

[G-W] K. Goodearl and R. Warfield *An Introduction to Noncommutative Noetherian Rings*, London Math. Soc. student texts, No. 15, Cambridge University Press, Cambridge, New York, 1989.

[L] T.Y. Lam *Lectures on Modules and Rings*, Graduate Texts in Mathematics, No. 189, Springer-Verlag, Berlin, Heidelberg, New York, 1998.

[M] I.G. Macdonald *Secondary representation of modules over a commutative ring*, in Symposia Mathematica, 11, p. 23-43, c. 1973.

[M-E] S. McAdam and P. Eakin *The Asymptotic Ass*, Journal of Algebra, 61, p. 71-81, c. 1979.

[Me] L. Melkersson *Content and inverse polynomials on artinian modules*, Communications in Algebra, 26 (4), p.1141-1145, c. 1998.