1. Consider an ordinary 52-card deck with 4 suits and 13 denominations.

   (a) In how many ways can a 5 card poker hand be drawn with no aces?

   Solution. There are 48 cards that are not aces, so there are $C(48,5) = \binom{48}{5} = \frac{48!}{43!\cdot 5!}$ ways to choose 5 cards with no aces. □

   (b) What is the probability of drawing four of a kind in a 5 card poker hand, that is, four cards of the same denomination?

   Solution. For each denomination there is one way to choose 4 cards of this denomination and 48 ways to choose the remaining card. There are 13 denominations. Thus there are $\mathbf{13 \cdot 48 = 624}$ ways to draw four of a kind. The probability of drawing 4 of a kind is the number of ways this can happen divided by the total number of hands, i.e. $\frac{\mathbf{13 \cdot 48}}{\mathbf{C(52,5)}}$. □

   (c) Suppose only two cards are drawn. What is the probability of drawing a two-card "blackjack," i.e. one ace and one of 10, J, Q, K?

   Solution. There are $52 \cdot 51$ ways to draw two cards from the deck (in order). There are 4 ways to draw an ace on the first card, and 16 ways to draw 10, J, Q or K on the second card, so there are $4 \cdot 16$ ways to draw an ace and then a 10, J, Q or K, and likewise for drawing the ace as the second card. Thus there are $2 \cdot 4 \cdot 16$ ways to draw a blackjack. Thus, the probability of drawing a blackjack is $\frac{\mathbf{2 \cdot 4 \cdot 16}}{\mathbf{52 \cdot 51}} = \frac{128}{2652} = .0483$. That is, you have a little less than a 1/20 chance of drawing a two-card blackjack. □

2. (a) Suppose \$100 is invested in a bank returning 12% interest compounded annually. How long will it take for the initial amount to double?

   Solution. The amount of after n years is $A_n = 1.12^n\$100$, so to find out how long it takes to double, we need to solve $A_n = 2 \cdot \$100$. Thus,

   $$1.12^n\$100 = 2 \cdot \$100 \quad \Rightarrow \quad 1.12^n = 2 \quad \Rightarrow \quad n\log(1.12) = \log(2)$$

   Thus, after $\mathbf{n} = \frac{\log(\mathbf{2})}{\log(\mathbf{1.12})} \approx 6.12$ years the initial amount will double. (Actually, we should take the ceiling, 7 years, since the interest is only compounded once per year.) □

   (b) What if the interest is compounded quarterly (4 times a year)?

   Solution. In this case, after 1 quarter there are $\left(1 + \frac{.12}{4}\right) A_0 = 1.03 \cdot A_0$ dollars in the account, after 2 quarters there are $1.03 \cdot 1.03 \cdot A_0 = 1.03^2 A_0$, etc., and after 4 quarters there are $1.03^4 A_0$ dollars in the account. Thus, after 1 year we have $1.03^4 A_0$ dollars, hence after n years we have $A_n = 1.03^{4n}\$100$. To find the time to double, we solve $A_n = 2 \cdot \$100$, or

   $$1.03^{4n}\$100 = 2 \cdot \$100 \quad \Rightarrow \quad 1.03^{4n} = 2 \quad \Rightarrow \quad 4n\log(1.03) = \log(2).$$

   Thus, after $\mathbf{n} = \frac{\log(\mathbf{2})}{\mathbf{4}\log(\mathbf{1.03})} \approx 5.86$ years the initial amount will double. Since the compounding is quarterly, the amount will double in 6 years. □

3. A batch of computers is ordered from 2 suppliers, A and B. 5% of the computers come from supplier A and the remaining 95% come from supplier B. It turns out that 70% of the computers from supplier A are defective, while only 10% of those from supplier B are defective. If a computer is selected at random and it turns out to be defective, which is more likely, that it came from supplier A or supplier B? (Hint: Find $P(A|\text{defective})$.)

Solution. Let A be the event that a computer is from supplier A, and def be the event that a computer is defective. We need to calculate the probability that a computer is from A, given that it is defective, that is to say, $P(A|def)$. We use Bayes' rule to compute

$$\begin{aligned} P(A|def) &= \frac{P(def|A)P(A)}{P(def|A)P(A) + P(def|\overline{A})P(\overline{A})} \\ &= \frac{0.7 \cdot 0.05}{0.7 \cdot 0.05 + 0.1 \cdot 0.95} \\ &= \frac{.035}{.035 + .095} = \frac{.035}{.13} = \frac{35}{130} \end{aligned}$$

Since $\frac{35}{130}$ is less than 1/2, we conclude that it is more likely that the defective computer came from supplier **B**. □

4. Suppose fruit flies are introduced into Orange County at time $n = 0$, so that the population at time 0 is 100 (measured in thousands). In the first year the population increases by 90, so that by year 1 the population is 190. Thereafter the growth in each succeeding year is 1/3 the growth in the preceding year, so that the population grows by 30 in year 2, by 10 in year 3, etc.

   (a) Write a recurrence relation for the population of fruit flies.

   Solution. Let a_n denote the number (in thousands) of fruit flies after n years. Then

   $$a_n - a_{n-1} = \frac{1}{3}\left(a_{n-1} - a_{n-2}\right), \qquad a_0 = 100,\ a_1 = 90.$$

   We can write this relation as $a_n - \frac{4}{3}a_{n-1} + \frac{1}{3}a_{n-2} = 0$. □

   (b) Solve the recurrence relation.

   Solution. We try a solution of the form $a_n = t^n$. Substituting into the recurrence relation, we have

   $$\begin{aligned} & t^n - \frac{4}{3}t^{n-1} + \frac{1}{3}t^{n-2} = 0 \\ \text{divide by } t^{n-2}: \quad & t^2 - \frac{4}{3}t + \frac{1}{3} = 0 \\ \text{multiply by } 3: \quad & 3t^2 - 4t + 1 = 0 \\ \text{factor terms}: \quad & (3t - 1)(t - 1) = 0 \end{aligned}$$

   This has solutions $t = 1/3$ and $s = 1$, so the general solution is $a_n = a1^n + b\left(\frac{1}{3}\right)^n = a + b\left(\frac{1}{3}\right)^n$, where a and b are any constants. We use the initial conditions to calculate a and b:

   $$a_0 = a + b(1/3)^0 = a + b = 100, \qquad \text{and} \qquad a_1 = a + b(1/3) = 190.$$

   This has the solution $a = 235$ and $b = -135$, so the solution to the recurrence relation is

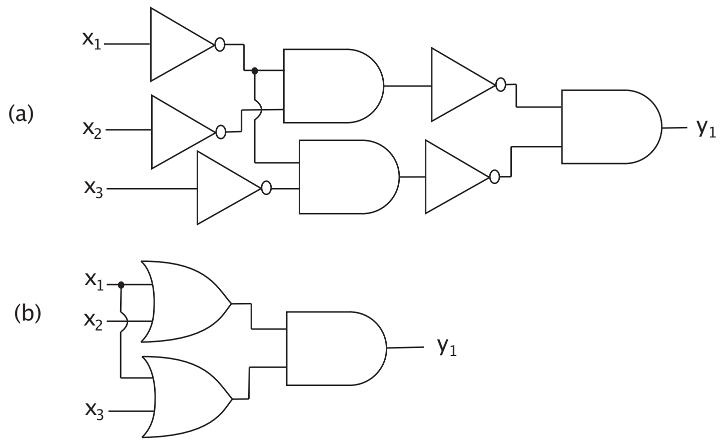
   $$a_n = 235 - 135\left(\frac{1}{3}\right)^n.$$

   □

(c) What happens to the population in the long run?

Solution. Since $1/3 < 1$, the term $\left(\frac{1}{3}\right)^n$ tends toward zero as n increases, so in the long run the population tends toward **235**. That is, after many years there will be 235,000 fruit flies in Orange County. $\square$

5. Write the Boolean expressions representing the following combinatorial circuits and show that they are equivalent.



$$\begin{array}{ll} \text{(a)} & (\overline{\overline{x_1} \wedge \overline{x_2}}) \wedge (\overline{\overline{x_1} \wedge \overline{x_3}}) \\ \text{(b)} & (x_1 \vee x_2) \wedge (x_1 \vee x_3) \end{array}$$

By de Morgan's law, we can transform the expression (a)

$$(\overline{\overline{x_1} \wedge \overline{x_2}}) \wedge (\overline{\overline{x_1} \wedge \overline{x_3}}) = (\overline{\overline{x_1}} \vee \overline{\overline{x_2}}) \wedge (\overline{\overline{x_1}} \vee \overline{\overline{x_3}}) = (x_1 \vee x_2) \wedge (x_1 \vee x_3),$$

which is exactly the expression (b).

(BONUS) Let f be a one-to-one function from $X = \{1, 2, \ldots, n\}$ onto X. Let

$$f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$$

denote the k-fold composition of f with itself.

(a) Show that for each $x \in X$ there is some j such that $f^j(x) = x$.

Solution. Let x be an arbitrary element of X. Now consider the set of numbers $\{f^i(x)\}_{i=1}^{n+1}$. In other words, the set of numbers obtained by applying f to x up to $n+1$ times. This is a set of $n+1$ numbers. Since $f^i(x)$ can only take on at most n values, by the pigeonhole principle, two of these must be the same. In other words, for some $i \neq j$, it must be that

$$f^i(x) = f^j(x).$$

Suppose that j is the smaller of i and j, and apply the inverse of f to both sides of the above equation j times:

$$\underbrace{f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}}_{j \text{ times}} \circ f^i(x) = \underbrace{f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}}_{j \text{ times}} \circ f^j(x)$$

This is just $f^{i-j}(x) = f^0(x)$. And since $f^0(x)$ is the result of applying f to x zero times, we have $f^k(x) = x$, where $k = i - j$. □

(b) Show that there are positive integers i and j such that $f^i(x) = f^j(x)$ for all $x \in X$.

Solution. Now consider $f^i = (f^i(1), f^i(2), \ldots, f^i(n))$, the set of elements of f^i applied to each element of X. Since f is one-to-one, f^i is a permutation of the numbers $1, 2, \ldots, n$. Now, there are $n!$ permutations of X, so consider the set $f, f^2, \ldots, f^{n!+1}$. This is a set of $n! + 1$ permutations. Thus, by the pigeonhole principle, at least two of these must be equal, i.e. let the permuations $f, \ldots, f^{n!+1}$ be the pigeons and the permutations of X be the holes. This shows that there are $i \neq j$ such that $f^i = f^j$, which means $f^i(x) = f^j(x)$ for all $x \in X$. □

(c) Show that for some positive integer k, $f^k(x) = x$ for all $x \in X$.

Solution. Since $f^i(x) = f^j(x)$ for all x, we may apply f^{-1} to both sides j times to obtain

$$f^{i-j}(x) = x,$$

or $f^k(x) = x$ for all x, where $k = i - j$. □