Math 489AB *Exercises for Chapter 0* *Fall 2008*

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 3 & 0 & -1 \\ 2 & -1 & 1 & 1 \end{bmatrix}.$$

(a) Find the row reduced echelon form of A.

Solution.

$$\begin{bmatrix} 1 & 0 & 0 & 1/15 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & 7/15 \end{bmatrix}$$

$\square$

(b) Find the general solution to the system of equations $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Solution. The row reduced echelon form of the augmented matrix $[A \,|\, \mathbf{b}]$ is

$$\begin{bmatrix} 1 & 0 & 0 & 1/15 & 3/5 \\ 0 & 1 & 0 & -2/5 & -3/5 \\ 0 & 0 & 1 & 7/15 & 1/5 \end{bmatrix},$$

so the general solution is

$$x_1 = \frac{3}{5} - \frac{1}{15}x_4, \qquad x_2 = -\frac{3}{5} + \frac{2}{5}x_4, \qquad x_3 = \frac{1}{5} - \frac{7}{15}x_4,$$

where x_4 is arbitrary. $\square$

(c) Find bases for the null space and range of A.

Solution. The nullspace is $\operatorname{span}[-1, 6, -7, 15]$. Since the row reduced echelon form of A has three nonzero rows, the range is all of $\mathbb{R}^3$. A basis for the range is $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. $\square$

(d) Find the rank of A and show that rank A + dimension of nullspace of $A = 4$.

Solution. The rank of A is 3 and the dimension of the null space is 1, so 3+1=4. $\square$

2. Prove the Cauchy-Schwarz Inequality:

$$|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|.$$

(Hint: For any number a, $0 \leq \|a\mathbf{x} + \mathbf{y}\|^2 = \langle a\mathbf{x} + \mathbf{y}, a\mathbf{x} + \mathbf{y}\rangle$. Expand this inequality and set $a = -\overline{\langle \mathbf{x}, \mathbf{y}\rangle}/\|\mathbf{x}\|^2$.)

Solution. If either $\mathbf{x}$ or $\mathbf{y}$ are zero, the inequality holds. So suppose neither are zero. Set a as above and calculate.

$$\begin{aligned}
0 &\leq \|a\mathbf{x} + \mathbf{y}\|^2 = \langle a\mathbf{x} + \mathbf{y}, a\mathbf{x} + \mathbf{y}\rangle \\
&= a\langle \mathbf{x}, a\mathbf{x} + \mathbf{y}\rangle + \langle \mathbf{y}, a\mathbf{x} + \mathbf{y}\rangle \\
&= a\overline{\langle a\mathbf{x} + \mathbf{y}, \mathbf{x}\rangle} + \overline{\langle a\mathbf{x} + \mathbf{y}, \mathbf{y}\rangle} \\
&= a\left(\overline{a}\|\mathbf{x}\|^2 + \langle \mathbf{x}, \mathbf{y}\rangle\right) + \overline{a}\overline{\langle \mathbf{x}, \mathbf{y}\rangle} + \|\mathbf{y}\|^2 \\
&= \underbrace{a\left(\overline{-\frac{\overline{\langle \mathbf{x}, \mathbf{y}\rangle}}{\|\mathbf{x}\|^2}}\|\mathbf{x}\|^2 + \langle \mathbf{x}, \mathbf{y}\rangle\right)}_{=0} + \overline{a}\overline{\langle \mathbf{x}, \mathbf{y}\rangle} + \|\mathbf{y}\|^2 \\
&= \overline{a}\overline{\langle \mathbf{x}, \mathbf{y}\rangle} + \|\mathbf{y}\|^2 \\
&= -\frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|^2}\overline{\langle \mathbf{x}, \mathbf{y}\rangle} + \|\mathbf{y}\|^2 \\
&= -\frac{|\langle \mathbf{x}, \mathbf{y}\rangle|}{\|\mathbf{x}\|^2} + \|\mathbf{y}\|^2
\end{aligned}$$

Add $|\langle \mathbf{x}, \mathbf{y}\rangle| / \|\mathbf{x}\|^2$ to both sides of the inequality and multiply by $\|\mathbf{x}\|^2$. ☐

3. Prove that if some basis of the vector space V consists of a finite number of elements, then all bases have the same number of elements.

Solution. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ are both bases for V and $m < n$. We will derive a contradiction. Since both are bases they span V, thus, for each j there are constants a_{ij} such that

$$\mathbf{w}_j = a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + \cdots + a_{nj}\mathbf{v}_m = \sum_{i=1}^{m} a_{ij}\mathbf{v}_i.$$

Consider the linear combination

$$\begin{aligned}
c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n &= \sum_{j=1}^{n} c_j\mathbf{w}_j \\
&= \sum_{j=1}^{n} c_j \sum_{i=1}^{m} a_{ij}\mathbf{v}_i \\
&= \sum_{i=1}^{m} \mathbf{v}_i \sum_{j=1}^{n} a_{ij}c_j
\end{aligned}$$

Now consider the set of equations

$$\sum_{j=1}^{n} a_{ij} c_j = 0, \qquad i = 1, 2, \ldots, m.$$

This is a set of m equations in n unknowns. Since $n > m$ the number of unknowns is greater than the number of equations, so there is a solution $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ such that not all of the c_j's are zero. But this implies that $c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n = 0$ where not all of the c_j's are zero, which contradicts the assumption that $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a basis and hence linearly independent. ☐

4. Prove the reverse-order laws: $(AB)^T = B^T A^T$ and $(AB)^* = B^* A^*$.

Solution. The second law follows from the first since $\overline{ab} = \overline{a}\overline{b}$ for any complex numbers a and b. So we prove the first. Consider the (i, j)th elements of the product AB and its transpose:

$$(AB)_{ij} = \sum_k a_{ik} b_{kj} \qquad \Rightarrow \qquad ((AB)^T)_{ij} = \sum_k b_{ki} a_{jk}$$

where the sum runs over the number of columns of A (which must be the number of rows of B). Now consider the (i, j)th element of the product $B^T A^T$:

$$\left(B^T A^T\right)_{ij} = \sum_k b_{ki} a_{jk}.$$

A comparison of the two formulas shows that $(AB)^T = B^T A^T$. ☐

5. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the following three vectors:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

Solution. We apply the Gram-Schmidt orthonormalization process. Set $\mathbf{z}_1 = \mathbf{x}_1 / \|\mathbf{x}_1\| = \frac{1}{2}\mathbf{x}_1$. Then set

$$\begin{aligned} \mathbf{y}_2 &= \mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{z}_1 \rangle \mathbf{z}_1 \\ &= \frac{1}{2}[-1, 1, -1, 1] \\ \mathbf{z}_2 &= \frac{\mathbf{y}_2}{\|\mathbf{y}_2\|} \\ &= \frac{1}{2}[-1, 1, -1, 1] \end{aligned}$$

We have the first two vectors. For the third, set

$$\begin{aligned}
\mathbf{y}_3 &= \mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{z}_1 \rangle \mathbf{z}_1 - \langle \mathbf{x}_3, \mathbf{z}_2 \rangle \mathbf{z}_2 \\
&= \frac{1}{2}[-1, -1, 1, 1] \\
\mathbf{z}_3 &= \frac{\mathbf{y}_3}{\|\mathbf{y}_3\|} \\
&= \frac{1}{2}[-1, -1, 1, 1]
\end{aligned}$$

Thus an orthonormal basis for the span of the three vectors is $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$, where

$$\mathbf{z}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{z}_2 = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{z}_3 = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

$\square$

6. Let $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Find the coordinates of $\mathbf{x}$ in the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Solution. We need to find c_1, c_2, c_3 so that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Perhaps the easiest way to do it is to solve the matrix equation $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\mathbf{c} = \mathbf{x}$. This has the solution $c_1 = 7, c_2 = -5, c_3 = -3$, so the coordinates of $\mathbf{x}$ in this basis are $(7, -5, -3)$. $\square$