Section 3.2

3.2.3. Suppose $A \in M_n$ is nilpotent, so that $A^k = 0$ for some integer k. Then by Problem 1.1.6, the only eigenvalue of A is 0. Thus $A = SJS^{-1}$ where J is the Jordan form of A. Since 0 is the only eigenvalue of A, J must be of the form

$$J = \begin{bmatrix} J_{n_1}(0) & & & 0 \\ & J_{n_2}(0) & & \\ & & \ddots & \\ 0 & & & J_{n_k}(0) \end{bmatrix},$$

where $n_1 + \cdots + n_k = n$. Consider the powers of A:

$$A^q = \underbrace{SJS^{-1}SJS^{-1} \cdots SJS^{-1}}_{q \text{ times}} = SJ^qS^{-1}.$$

The powers of J are given by

$$J^q = \begin{bmatrix} J_{n_1}(0)^q & & & 0 \\ & J_{n_2}(0)^q & & \\ & & \ddots & \\ 0 & & & J_{n_k}(0)^q \end{bmatrix}.$$

Now, $J_{n_i}(0)^{n_i} = 0$, so $J^r = A^r = 0$ where r is the maximum of the n_i's. Since $n_i \le n$, $A^r = 0$ for some $r \le n$. Note that $r < n$ unless J is itself a Jordan block, in which case $r = n$.

3.2.6. The linear transformation $p(t) \to p'(t)$ on the space of polynomials of degree at most 3 has the matrix representation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A is upper triangular with 0 diagonal entries, so the only eigenvalue of A is 0. To find the Jordan form of A we must determine the geometric multiplicity of the zero eigenvalue. Thus, we calculate the eigenvector(s) of A, which satisfy

$$A\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \quad \Rightarrow \quad x_2 = x_3 = x_4 = 0.$$

Thus, the only eigenvector of A is $\mathbf{x} = [1, 0, 0, 0]^T$. Since the dimension of the eigenspace of $\lambda = 0$ is 1, A has a single Jordan block. Thus the Jordan form of A is

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.2.8. If $p_A(t) = (t+3)^4(t-4)^2$, then A has eigenvalues -3 and 4, with multiplicities 4 and 2, respectively. Thus the Jordan form J of A contains Jordan blocks corresponding to -3 and 4 such that their sizes add up to 4 and 2. Thus J must be of the form

$$J = \begin{bmatrix} J_{n_1}(-3) & & & & & \mathbf{0} \\ & J_{n_2}(-3) & & & & \\ & & J_{n_3}(-3) & & & \\ & & & J_{n_4}(-3) & & \\ & & & & J_{m_1}(4) & \\ \mathbf{0} & & & & & J_{m_2}(4) \end{bmatrix},$$

where $n_1+n_2+n_3+n_4 = 4$ and $m_1+m_2 = 2$. (We allow blocks to be of size 0.) The sizes of the blocks for $\lambda = -3$ can be 4, 3 and 1, 2 and 2, 2 and 1 and 1, or 1 and 1 and 1 and 1, and the sizes of the blocks for $\lambda = 4$ can be 1 and 1 or 2. Thus there are $5 \times 2 = 10$ possibilities. These are:

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$\begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

WHEW!

3.2.9. Let $A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$. The characteristic polynomial of A is

$$p_A(t) = \det \begin{bmatrix} t-i & -1 \\ -1 & t+i \end{bmatrix} = (t-i)(t+i) - 1 = t^2 - i^2 - 1 = t^2.$$

Thus A has a single eigenvalue $\lambda = 0$, with multiplicity 2. We look for eigenvectors:

$$A\mathbf{x} = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ix_1 + x_2 \\ x_1 - ix_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Thus there is a single eigenvector for A, so the eigenspace for $\lambda = 0$ has dimension 1. This means there is a single Jordan block in the Jordan form, so the Jordan form must be

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Section 3.5

3.5.4. Let $A \in M_n$ be such that all leading principal minors are nonzero. Then by Theorem 3.5.5 A can be factored as $A = LU$. Since A is nonsingular the diagonal elements of L and U are nonzero, and one or the other of the diagonals may be chosen arbitrarily. We perform Gaussian elimination. To zero out the first column before the first row, subtract a_{i1}/a_{11} times row 1 from row i for rows $i = 2, \ldots, n$. Each of these operations is equivalent to multiplying by an elementary matrix of type 3 (see section 0.3.3). Note that the inverse of the type 3 elementary matrix is obtained by replacing c with $-c$. Consider the first action on

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Subtract a_{21}/a_{11} times row 1 from row 2.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_{21}/a_{11} & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & * & \cdots & * \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_{21}/a_{11} & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & * & \cdots & * \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

where the $*$'s represent terms resulting from the operation. Continue down the first column. Subtract a_{i1}/a_{11} times row 1 from row i for rows $i = 2, \ldots, n$. We obtain

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21}/a_{11} & 1 & 0 & \cdots & 0 \\ a_{31}/a_{11} & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ a_{n1}/a_{11} & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & & \\ 0 & * & \cdots & * \end{bmatrix}$$

Next proceed to the second, third, etc., columns. Subtract p_{ij} times row i from row j to zero out the element in position (i, j). This results in a factorization.

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ p_{21} & 1 & 0 & \cdots & 0 \\ p_{31} & p_{32} & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ p_{n1} & p_{n2} & p_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & * & \cdots & * \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & * \end{bmatrix} = LU$$

This process results in an LU factorization in which the diagonal elements of L are 1.

Chapter 4

Section 4.1

4.1.2. Suppose $A \in M_n$ is Hermitian, i.e. $A^* = A$. Then for any $S \in M_n$,

$$(SAS^*)^* = S^{**}A^*S^* = SAS^*,$$

so SAS^* is Hermitian.

Now consider SAS^{-1} for a nonsingular S. If S is unitary, i.e. $S^{-1} = S^*$, then by the previous argument $SAS^{-1} = SAS^*$ is Hermitian. But, generally

$$\left(SAS^{-1}\right)^* = \left(S^{-1}\right)^* AS^*.$$

This is generally not equal to SAS^{-1} unless S is unitary.

4.1.4. We verify the following properties:

1. $A + A^*$, AA^*, and A^*A are all Hermitian for all $A \in M_n$.

$$\begin{aligned} (A + A^*)^* &= A^* + A^{**} = A^* + A = A + A^* \\ (AA^*)^* &= A^{**}A^* = AA^* \\ (A^*A)^* &= A^*A^{**} = A^*A \end{aligned}$$

2. If A is Hermitian, then A^k is Hermitian for all $k = 1, 2, \ldots$. If A is nonsingular, then A^{-1} is Hermitian.

   For the first, take the Hermitian adjoint of the product A^k:

$$\left(A^k\right)^* = (AA\cdots A)^* = A^*A^*\cdots A^* = AA\cdots A = A^k.$$

   For the second, use the fact that the Hermitian adjoint of the identity is the identity:

$$I = A^{-1}A = AA^{-1} = \left(AA^{-1}\right)^* = \left(A^{-1}\right)^* A^* = \left(A^{-1}\right)^* A$$

   Multiplying $A^{-1}A = \left(A^{-1}\right)^* A$ on the right by A^{-1}, we have $\left(A^{-1}\right)^* = A^{-1}$

3. If A and B are Hermitian, then $\alpha A + \beta B$ is Hermitian for all real α, β.

$$(\alpha A + \beta B)^* = (\alpha A)^* + (\beta B)^* = \alpha A^* + \beta B^* = \alpha A + \beta B.$$

4. $A - A^*$ is skew-Hermitian for all $A \in M_n$.

$$(A - A^*)^* = A^* - A^{**} = A^* - A = -\left(A - A^*\right).$$

5. If A and B are skew-Hermitian, then $\alpha A + \beta B$ is skew-Hermitian for all real α, β.

$$(\alpha A + \beta B)^* = (\alpha A)^* + (\beta B)^* = \alpha A^* + \beta B^* = -\alpha A - \beta B = -(\alpha A + \beta B).$$

6. If A is Hermitian, then iA is skew-Hermitian.

$$(iA)^* = \bar{i}A^* = -iA.$$

7. If A is skew-Hermitian, then iA is Hermitian.

$$(iA)^* = \bar{i}A^* = \bar{i}(-A) = -i(-A) = iA.$$

8. Any $A \in M_n$ may be written as $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = H(A) + S(A)$, where $H(A) = \frac{1}{2}(A + A^*)$ is the *Hermitian part* of A, and $S(A) = \frac{1}{2}(A - A^*)$ is the *skew-Hermitian part* of A.

   The fact that $A = H(A)+S(A)$ follows immediately. The fact that $H(A)$ is Hermitian follows from properties 1 and 3, and the fact that $S(A)$ is skew Hermitian follows from properties 4 and 5.

9. If A is Hermitian, the main diagonal entries of A are all real. In order to specify the n^2 elements of A one may specify freely any n real numbers (for the main diagonal entries) and any $\frac{1}{2}n(n-1)$ complex numbers (for the off-diagonal elements).

   If A is Hermitian, $\overline{a_{ij}} = a_{ji}$. For the diagonal elements, this means that $\overline{a_{ii}} = a_{ii}$. Since a complex number is equal to its complex conjugate if and only if it is real, this is only possible if a_{ii} is real for all i.

   To form a Hermitian matrix A, we thus need n real numbers for the diagonal elements. This leaves $n^2 - n = n(n-1)$ off-diagonal elements. To satisfy the condition $\overline{a_{ij}} = a_{ji}$ we can choose 1/2 of these off-diagonal elements arbitrarily, then the others are determined by this condition.

4.1.6. Proposition: Every matrix $A \in M_n$ is uniquely determined by the Hermitian form $\mathbf{x}^*A\mathbf{x}$. I.e., let $A, B \in M_n$. Then, if $\mathbf{x}^*A\mathbf{x} = \mathbf{x}^*B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}^n$, then $A = B$.

Proof. Suppose $\mathbf{x}^*A\mathbf{x} = \mathbf{x}^*B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}^n$. Let $C = A - B$. Then

$$\mathbf{x}^*C\mathbf{x} = \mathbf{x}^*(A - B)\mathbf{x} = \mathbf{x}^*A\mathbf{x} - \mathbf{x}^*B\mathbf{x} = 0$$

for all $\mathbf{x} \in \mathbb{C}^n$. Now consider the form $(\mathbf{x} + \mathbf{y})^*C(\mathbf{x} + \mathbf{y})$:

$$\begin{aligned} 0 &= (\mathbf{x} + \mathbf{y})^*C(\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x}^*C(\mathbf{x} + \mathbf{y}) + \mathbf{y}^*C(\mathbf{x} + \mathbf{y}) = \mathbf{x}^*C\mathbf{x} + \mathbf{x}^*C\mathbf{y} + \mathbf{y}^*C\mathbf{x} + \mathbf{y}^*C\mathbf{y} \\ &= \mathbf{x}^*C\mathbf{y} + \mathbf{y}^*C\mathbf{x} \end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Now choose $\mathbf{x} = \mathbf{e}_k$ (the kth standard basis vector) and $\mathbf{y} = e^{i\theta}\mathbf{e}_j$. Then $\mathbf{x}^* = \mathbf{e}_k^*$ and $\mathbf{y}^* = e^{-i\theta}\mathbf{e}_j$. Notice that

$$\mathbf{x}^*C\mathbf{y} = e^{i\theta}\mathbf{e}_kC\mathbf{e}_j = e^{i\theta}\mathbf{e}_k\mathbf{c}_j = e^{i\theta}c_{kj},$$

where $\mathbf{c}_j$ is the jth column of C. Likewise,

$$\mathbf{y}^*C\mathbf{x} = e^{-i\theta}\mathbf{e}_jC\mathbf{e}_k = e^{-i\theta}\mathbf{e}_j\mathbf{c}_k = e^{i\theta}c_{jk}.$$

Therefore,

$$\begin{aligned} 0 &= \mathbf{x}^*C\mathbf{y} + \mathbf{y}^*C\mathbf{x} = e^{i\theta}c_{kj} + e^{-i\theta}c_{jk} \\ \Rightarrow \ e^{2i\theta}c_{kj} &= -c_{jk} \end{aligned}$$

for all real θ and all $j, k = 1, 2, \ldots, n$. Now, if we take $\theta = \pi/2$ in the above equation, we have $e^{2i\pi/2} = -1$, so

$$c_{kj} = c_{jk}.$$

That is, C is symmetric. Then, if we take $\theta = 0$ we have

$$c_{kj} + c_{jk} = 0 = 2c_{kj}.$$

This must hold for all j, k, so $C = 0 = A - B$, which implies that $A = B$. ∎

4.1.10. We have seen already (problem 4.1.4, part 6) that if A is Hermitian then iA is skew-Hermitian. Likewise, if iA is skew-Hermitian, then by part 7 of that same problem, $iiA = -A$ is Hermitian, and by part 3, $(-1)(-A) = A$ is Hermitian. Thus we have shown that A is Hermitian if and only if iA is skew-Hermitian.

Now, let $A \in M_n$ be skew-Hermitian. Then iA is Hermitian. Suppose λ is an eigenvalue of A, with associated eigenvector $\mathbf{x}$. Take $A\mathbf{x} = \lambda\mathbf{x}$ and multiply both sides by i:

$$iA\mathbf{x} = i\lambda\mathbf{x}.$$

Thus $i\lambda$ is an eigenvalue of iA. And since iA is Hermitian, $i\lambda$ must be real. But this is only possible if $\lambda = i\beta$ where β is real, that is, if λ is pure imaginary. Thus all the eigenvalues of a skew-Hermitian matrix are pure imaginary.

Consider the eigenvalues of A^2. By Theorem 1.1.6 the eigenvalues of A^2 are the squares of the eigenvalues of A. So if $i\beta$ is an eigenvalue of A, then $(i\beta)^2 = -\beta^2$ is an eigenvalue of A^2. Since all the eigenvalues of A have the form $i\beta$, the eigenvalues of A^2 are all of the form $-\beta^2$, where β is real. Thus the eigenvalues of A^2 are all real and non-positive.

4.1.12. Let $A \in M_n$ be Hermitian. Then A is similar to a diagonal matrix D with real entries, $A = SDS^{-1}$. Since rank is a similarity invariant, $\operatorname{rank}(A) = \operatorname{rank}(D)$. But $\operatorname{rank}(D)$ is obviously the number of non-zero entries on its diagonal, i.e. the number of non-zero eigenvalues of A.

This result is generally not true, for consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then A is upper triangular with zero diagonal entries so A has no non-zero eigenvalues. However, $\operatorname{rank}(A) = 1$, which is greater than the number of non-zero eigenvalues of A.

Section 4.2

4.2.2. Let $A \in M_n$ be Hermitian. Consider the three optimization problems:

(a) $\max_{\mathbf{x}^*\mathbf{x}=1} \mathbf{x}^* A\mathbf{x}$,

(b) $\max_{\mathbf{x}\neq 0} \dfrac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}}$, and

(c) $\max_{\mathbf{x}^* A\mathbf{x}=1} \dfrac{1}{\mathbf{x}^*\mathbf{x}}$ if at least one eigenvalue of A is positive.

We will show that (b) has the same solution as (a) and (c) has the same solution as (b). Notice that

$$\begin{aligned}\max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} &= \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^* A\mathbf{x}}{\|\mathbf{x}\|^2} = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^*}{\|\mathbf{x}\|} A \frac{\mathbf{x}}{\|\mathbf{x}\|} \\ &= \max_{\mathbf{x}\neq 0} \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)^* A \frac{\mathbf{x}}{\|\mathbf{x}\|}\end{aligned}$$

Now set $\mathbf{y} = \mathbf{x}/\|\mathbf{x}\|$ in the above expression, and note that $\|\mathbf{y}\| = 1$, so

$$\max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \max_{\mathbf{y}^*\mathbf{y}=1} \mathbf{y}^* A\mathbf{y}.$$

Thus (b) has the same solution as (a).

To show that (c) has the same solution as (b), note that

$$\begin{aligned}\max_{\mathbf{x}^* A\mathbf{x}=1} \frac{1}{\mathbf{x}^*\mathbf{x}} &= \max_{\mathbf{x}^* A\mathbf{x}=1} \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \\ &\leq \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\end{aligned}$$

Now consider the largest eigenvalue $\lambda_{\max}$ of A, with associated eigenvector $\mathbf{y}$. Then

$$\mathbf{y}^* A\mathbf{y} = \mathbf{y}^* \lambda_{\max}\mathbf{y} = \lambda_{\max}\mathbf{y}^*\mathbf{y}.$$

Take the eigenvector $\mathbf{y}$ so that $\mathbf{y}^* A\mathbf{y} = 1$. Then

$$\max_{\mathbf{x}^* A\mathbf{x}=1} \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \geq \frac{\mathbf{y}^* A\mathbf{y}}{\mathbf{y}^*\mathbf{y}} = \lambda_{\max}.$$

Since A is Hermitian, Theorem 4.2.2 implies that $\lambda_{\max} = \max_{\mathbf{x}\neq 0} \dfrac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}}$. Combine the above two inequalities:

$$\max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \leq \max_{\mathbf{x}^* A\mathbf{x}=1} \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \leq \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}}$$

Thus $\max_{\mathbf{x}^* A\mathbf{x}=1} \dfrac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \max_{\mathbf{x}\neq 0} \dfrac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}}$, so (c) has the same solution as (b).

4.2.3. If $A \in M_n$ is Hermitian and $\mathbf{x}^*\mathbf{x} = 1$, Theorem 4.2.2 implies that

$$\lambda_{\max} = \max_{\mathbf{y}^*\mathbf{y}=1} \mathbf{y}^* A\mathbf{y} \geq \mathbf{x}^* A\mathbf{x} \geq \min_{\mathbf{y}^*\mathbf{y}=1} \mathbf{y}^* A\mathbf{y} = \lambda_{\min}$$

4.2.4 To show that the assumption that A is Hermitian is essential in Theorem 4.2.2, consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Then

$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = x_1^2 + 2x_1x_2 + x_2^2$$

Thus,

$$\begin{aligned} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} &= \frac{x_1^2 + 2x_1x_2 + x_2^2}{x_1^2 + x_2^2} = \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} + \frac{2x_1x_2}{x_1^2 + x_2^2} \\ &= 1 + \frac{2x_1x_2}{x_1^2 + x_2^2} \end{aligned}$$

If we take, for example, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\dfrac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = 1 + 2/(1+1) = 2$. Therefore,

$$\max_{0 \neq \mathbf{x} \in \mathbb{C}^2} \frac{\mathbf{x}^* A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \geq \max_{0 \neq \mathbf{x} \in \mathbb{R}^2} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \geq 2.$$

However, the maximal eigenvalue of A is 1, so Theorem 4.2.2 does not apply to the non-Hermitian matrices.