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On the eigenvalues of double band matrices

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ABSTRACT

We consider matrices containing two diagonal bands of positive entries. We show that all eigenvalues of such matrices are of the form $r\zeta$, where r is a nonnegative real number and ζ is a pth root of unity, where p is the period of the matrix, which is computed from the distance between the bands. We also present a problem in the asymptotics of spectra in which such double band matrices are perturbed by banded matrices.

1. Introduction

Let b and k be positive integers and consider the double band matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & a_{1,k+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_{2,k+2} & \cdots & 0 \\ \vdots & & & & & \ddots & \\ a_{b+1,1} & 0 & \cdots & & & & \\ 0 & a_{b+2,2} & \cdots & & & & \\ & & \ddots & & & & \end{pmatrix}, \tag{1}$$

where the diagonals $a_{b+j,j}$, $a_{i,k+i}$ are positive real numbers and the remaining entries are zero. Such matrices arise, for example, in second order differential equations, such as the generalized Lamé equation [6]. The special case when A is Toeplitz has been studied for a long time, particularly in

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the asymptotic limit as the size of the matrix becomes unbounded. The location of the eigenvalues in this limit are calculated, for example, in [2,3]. Curiously, even in this restricted case, the location of the eigenvalues of finite Toeplitz matrices has not been described, as far as we are aware. Such a description is provided as a special case of our main result below.

The matrices as in (1) have a period p, which is the greatest common divisor of the lengths of cycles in the directed graph $\Gamma(A)$.[1] Thus it is given by

$$p = \frac{b+k}{\gcd(b,k)}. \tag{2}$$

If b and k are relatively prime then A is irreducible. The Perron–Frobenius theorems [5] tell us that the spectrum of a nonnegative and irreducible matrix with period p is invariant under rotations by $2\pi/p$ and that there are exactly p eigenvalues with maximal modulus $\rho(A)$ given by $\rho(A)\exp(i2\pi j/p)$, $j = 0,\ldots,p-1$. The main result of this paper is to show that for matrices with the special form (1), this holds generally, and that, in addition to this, the eigenvalues all lie on the lines through the pth roots of unity in the complex plane.

Theorem 1. *Let A be an $n \times n$ matrix as in (1). Let $n = mp + q$ for $0 \leqslant q < p$, and let $g = \gcd(b,k)$. If $g = 1$ then A has a zero eigenvalue of multiplicity q, and mp eigenvalues*

$$r_s\, e^{i2\pi j/p}, \quad j = 0,\ldots,p-1,\ s = 1,\ldots,m, \tag{3}$$

where r_s, $s = 1,\ldots,m$, are distinct, real and positive.

More generally, the nonzero eigenvalues of A are given by

$$r_{s,t}\, e^{i2\pi j/p}, \quad j = 0,\ldots,p-1,\ t = 1,\ldots,g,\ s = 1,\ldots,\left\lfloor \frac{\left\lfloor \frac{n-t}{g} \right\rfloor + 1}{p} \right\rfloor, \tag{4}$$

where for each fixed t, the $r_{s,t}$'s are distinct, real and positive. The zero eigenvalue of A has multiplicity

$$g\left(\left\lfloor \frac{n}{g} \right\rfloor \bmod p\right) + (n \bmod g)\left[\left(\left\lfloor \frac{n}{g} \right\rfloor + 1\right) \bmod p - \left\lfloor \frac{n}{g} \right\rfloor \bmod p\right]. \tag{5}$$

Before proceeding to the proof we make a few remarks. First, if $b = k$, i.e. the bands are an equal distance from the diagonal, then $p = 2$, so all the eigenvalues are real, and come in positive and negative pairs. The case $b = 1$, $k = 2$ was considered in [6], where it was shown that all eigenvalues are of the form $r\exp(i2\pi j/3)$. Theorem 1 is a generalization of the result in that paper. The following corollaries are immediate consequences of the construction of the proof of the above theorem, and generalizes to the case when the two bands have opposite signs or contain nonnegative or complex entries. Note that by $|A|$ we mean the matrix of absolute values of A, i.e. $|A| = \{|a_{ij}|\}_{ij}$.

Corollary 1. *Let A be a double band matrix such that the entries of one diagonal are positive and the entries of the other are negative. Then the eigenvalues of A are obtained from the eigenvalues of $|A|$ by rotating them by $\pi b/(b+k)$ if the $a_{i,k+i}$ entries are negative, and by $-\pi k/(b+k)$ if the $a_{b+j,j}$ entries are negative. Therefore, the eigenvalues are all of the form*

$$r\exp\left[i\left(2\pi j/p + \frac{\pi b}{b+k}\right)\right] \quad or \quad r\exp\left[i\left(2\pi j/p - \frac{\pi k}{b+k}\right)\right], \tag{6}$$

depending on which diagonal is negative. The multiplicities of the eigenvalues are as described in Theorem 1.

[1] The period is also sometimes referred to as the *index*, or *index of imprimitivity*. A primitive matrix is one in which $p = 1$. A nonnegative matrix A induces a directed graph $\Gamma(A)$ on n nodes by assigning an arc between nodes i and j if the a_{ij} entry is nonzero. A cycle is a succession of arcs that begins and ends at the same node. See [7] for more on the relationship between directed graphs and nonnegative matrices.

Corollary 2. *Let A be an $n \times n$ complex matrix with $a_{ij} = 0$ for all $i - j \neq b$ or $-k$. Then the spectrum of A is invariant under rotations by $2\pi/p$. If, in addition, A is real and nonnegative, then the eigenvalues of A are all of the form $r\,e^{i2\pi j/p}$, where r is a nonnegative real number.*

In the following section we prove the above theorem and corollaries. Following this we discuss an asymptotic problem. We consider the case in which such double band matrices are perturbed by banded matrices. Can we say anything about the spectra in this case?

2. Proofs of main results

We first prove the following lemma that allows us to reduce the problem to the case when b and k are relatively prime. If $g > 1$ we can move the diagonals "inward" at the expense of rearranging them and inserting zeros. In the proofs that follow we will employ many of the results from the theory of nonnegative matrices, which can be found in [7,5].

Lemma 1. *A is cogredient*[2] *to a direct sum of g matrices of the form (1) where the b and k are relatively prime. That is, there is a permutation matrix P such that*

$$B = PAP^T = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & B_g \end{pmatrix}, \tag{7}$$

where B_i is an $(\lfloor \frac{n-i}{g} \rfloor + 1) \times (\lfloor \frac{n-i}{g} \rfloor + 1)$ square matrix with nonzero entries at the positions $(n_b + j, j)$ and $(i, n_k + i)$ and zeros everywhere else, where $n_b = b/g$ and $n_k = k/g$.

Proof. We construct P explicitly. Let σ be the permutation of the integers $1, \ldots, n$ defined as follows. For $i = 1, \ldots, g$, define

$$n_i = \left\lfloor \frac{n-i}{g} \right\rfloor + 1 \tag{8}$$

and the partial sums $N_1 = 0$, $N_i = n_1 + \cdots + n_{i-1}$ for $i = 2, \ldots g$. A simple calculation shows that $n_1 + n_2 + \cdots + n_g = n$. Now, for each $i = 1, \ldots, g$ define

$$\sigma_{N_i + j} = i + (j-1)g, \quad j = 1, 2, \ldots, n_i. \tag{9}$$

Then σ is a permutation of $(1, \ldots, n)$, and we have

$$\sigma_{n_b + j} - \sigma_j = b, \tag{10}$$
$$\sigma_{n_k + i} - \sigma_i = k, \tag{11}$$

for all pairs $N_i < n_b + j, j \leqslant N_i + n_i$ for some i, and likewise for $n_k + i, i$. Now we define the permutation matrix P by setting $p_{j,\sigma_j} = 1$ for $j = 1, \ldots, n$. Then, define $B = PAP^T$, so that the (i,j)th entry of B is $b_{ij} = a_{\sigma_i, \sigma_j}$. Since $a_{ij} \neq 0$ if and only if $i - j = b$ or $i - j = -k$, as long as $N_i < n_b + j, j \leqslant N_i + n_i$ for some i and $N_l < n_k + i, i \leqslant N_l + n_l$ for some l, we have

$$b_{n_b + j, j} = a_{\sigma_{n_b + j}, \sigma_j} = a_{s+b, s}, \tag{12}$$
$$b_{i, n_k + i} = a_{\sigma_i, \sigma_{n_k + i}} = a_{t, t+k}, \tag{13}$$

for some s and t.

Notice that each square block $N_l < i, j \leqslant N_l + n_l$ on the diagonal of B contains $n_l - n_b$ nonzero elements at the positions $(n_b + j, j)$ and $n_l - n_k$ nonzero elements in the positions $(i, n_k + i)$, for a

[2] A matrix A is cogredient to B if there is a permutation matrix P such that $A = PBP^T$.

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total of $n - gn_b = n - b$ elements in the positions $(n_b + j, j)$ and $n - gn_k = n - k$ elements in the positions $(i, n_k + i)$. Thus, since PAP^T moves each element of A to an unique position, we see that B has the form (7), as required. □

We can now proceed to the proof of Theorem 1.

Proof of Theorem 1. Suppose, first of all, that b and k are relatively prime. Then $p = b + k$. Also, we may assume that $b \leqslant k$ without loss of generality. Since A is irreducible with period p it is cogredient to a matrix in superdiagonal block form. That is, there is a permutation matrix P such that

$$C = PAP^T = \begin{pmatrix} 0 & C_1 & 0 & \cdots & 0 \\ 0 & 0 & C_2 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & C_{p-1} \\ C_p & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{14}$$

where the diagonal zero blocks are square, but the blocks C_j may be rectangular. The key to the proof is that we can find a P such that the matrices C_j are bidiagonal. We construct P explicitly. We first construct a permutation σ of the integers from 1 to n. For each $i = 0, \ldots, p-1$ define γ_i, z_i, and n_i by

$$\gamma_i = (ik+1) \bmod p = (ik+1) + z_i, \tag{15}$$

$$n_i = \left\lfloor \frac{n - \gamma_i}{p} \right\rfloor + 1, \tag{16}$$

and $n_{-1} = 0$. An elementary calculation shows that $n_0 + \cdots + n_{p-1} = n$. To simplify notation we also define the partial sums $N_i = n_0 + \cdots + n_i$. Now, for each $i = 0, \ldots, p-1$, define

$$\sigma_{N_{i-1}+j} = \gamma_i + (j-1)p, \quad j = 1, \ldots, n_i. \tag{17}$$

Thus $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a permutation of the numbers $(1, \ldots, n)$.

Now we define the permutation matrix $p_{j,\sigma(j)} = 1$. Then the (i,j) entry of $C = PAP^T$ is $c_{ij} = a_{\sigma_i,\sigma_j}$. For ordered subsets α, β of $\{1, \ldots, n\}$, let $A(\alpha, \beta)$ be the submatrix with rows in α and columns in β. We define the submatrices C_i as follows:

$$C_i = C\left(\{N_{i-2}+1, \ldots, N_{i-2}+n_{i-1}\}, \{N_{i-1}+1, \ldots, N_{i-1}+n_i\}\right), \quad i = 1, \ldots, p-1, \tag{18}$$

$$C_p = C\left(\{N_{p-2}+1, \ldots, n\}, \{1, \ldots, n_0\}\right). \tag{19}$$

Note that since b and k are relatively prime, and $b \leqslant k$, $z_i - z_{i+1}$ is either 0 or 1. We now use this fact to show that each C_i is bidiagonal.

Consider the main, lower and upper diagonals of C_i for $i = 1, \ldots, p-1$. Note that

$$\begin{aligned} \sigma_{N_{i-1}+j} - \sigma_{N_i+j} &= \gamma_i + (j-1)p - (\gamma_{i+1} + (j-1)p) \\ &= -k + (z_i - z_{i+1})\,p \\ &= \begin{cases} -k & \text{if } z_i - z_{i+1} = 0, \\ b & \text{if } z_i - z_{i+1} = 1, \end{cases} \end{aligned} \tag{20}$$

$$\begin{aligned} \sigma_{N_{i-1}+j+1} - \sigma_{N_i+j} &= \gamma_i + jp - (\gamma_{i+1} + (j-1)p) \\ &= b + (z_i - z_{i+1})\,p \\ &= \begin{cases} b & \text{if } z_i - z_{i+1} = 0, \\ b+p & \text{if } z_i - z_{i+1} = 1, \end{cases} \end{aligned} \tag{21}$$

$$\begin{aligned} \sigma_{N_{i-1}+j} - \sigma_{N_i+j+1} &= \gamma_i + (j-1)p - (\gamma_{i+1} + jp) \\ &= -k + (z_i - z_{i+1})\,p - p \\ &= \begin{cases} -k-p & \text{if } z_i - z_{i+1} = 0, \\ -k & \text{if } z_i - z_{i+1} = 1. \end{cases} \end{aligned} \tag{22}$$

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Now, since $a_{i,j} \neq 0$ if and only if $i - j = b$ or $i - j = -k$, we see that each $C_i, i = 1, \ldots, p-1$, is upper bidiagonal if $z_{i-1} - z_i = 1$ and lower bidiagonal if $z_{i-1} - z_i = 0$, and that the entries in the two nonzero diagonals are all positive.

Next, consider C_p. We have

$$\begin{aligned} \sigma_{N_{p-2}+j} - \sigma_j &= \gamma_{p-1} + (j-1)p - (1 + (j-1)p) \\ &= b, \end{aligned} \tag{23}$$

$$\begin{aligned} \sigma_{N_{p-2}+j} - \sigma_{j+1} &= \gamma_{p-1} + (j-1)p - (1 + jp) \\ &= -k. \end{aligned} \tag{24}$$

Thus C_p is an upper bidiagonal $n_{p-1} \times n_0$ matrix.

Now we have C in the form (14). Thus

$$C^p = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & 0 & D_p \end{pmatrix}, \tag{25}$$

where

$$D_1 = C_1 C_2 \cdots C_p,\ D_2 = C_2 C_3 \cdots C_p C_1,\ \ldots,\ D_p = C_p C_1 \cdots C_{p-1}. \tag{26}$$

It follows that the nonzero spectra of each of the products D_j are all equal and that the nonzero spectra of C consists of the pth roots of the eigenvalues of D_j. Notice that each of the D_j are square $n_{j-1} \times n_{j-1}$ matrices. Furthermore, for each j, $n_j = m = \lfloor n/p \rfloor$ or $n_j = m + 1$, and for at least one j, $n_j = m$. Thus, if $m = 0$ all the eigenvalues are zero. So we assume that $m > 0$.

So let j be such that $n_{j-1} = m$, so that D_j is $m \times m$. We will show that the eigenvalues of D_j are all real, positive and distinct. For this it is sufficient to show that D_j is oscillatory. First, we recall a few definitions. A matrix is totally nonnegative (positive) if all minors of all sizes are nonnegative (positive). A matrix X is oscillatory if there exists a positive integer s such that X^s is totally positive. Oscillatory matrices have the remarkable property that their eigenvalues are distinct, real and positive [7, cf. Chapter 6, Theorem 3.2]. Moreover, a matrix X is oscillatory if and only if it is (a) totally nonnegative, (b) nonsingular, and (c) satisfies $x_{i,i+1}, x_{i+1,i} > 0$ for all i. We will verify these three conditions for D_j.

Condition (c) follows from the fact that $z_0 = z_1 = 0$, so C_1 is lower bidiagonal. And since C_p is upper bidiagonal, D_j is the product of upper and lower bidiagonal matrices. Thus, the super and sub-diagonal entries of D_j are all positive. Conditions (a) and (b) follow from the Cauchy–Binet identity. For any pair of ordered subsets $\alpha, \beta \subset \{1, 2, \ldots, m\}$ of the same cardinality, the determinant of $D_j(\alpha, \beta)$ is

$$\begin{aligned} \det\left(D_j(\alpha,\beta)\right) &= \det\left(\left(C_j C_{j+1} \cdots C_{[[j+p-1]]}\right)(\alpha,\beta)\right) \\ &= \sum_{\theta_j,\ldots,\theta_{j+p-2}} \prod_{i=j}^{(j+p-1)} \det\left(C_{[[i]]}\left(\theta_{i-1},\theta_i\right)\right), \end{aligned} \tag{27}$$

where $[[i]] = i \bmod p$, $\theta_{j-1} = \alpha, \theta_{j+p-1} = \beta$, and the sum is taken over all ordered subsets $\theta_j, \ldots, \theta_{j+p-2}$ where the submatrices are defined. A straightforward calculation shows that all minors of bidiagonal matrices with positive entries on the two diagonals are nonnegative. It immediately follows from (27) that D_j is totally nonnegative. Thus, condition (a) is satisfied.

It remains to be shown that D_j is nonsingular. For this we use (27) again, with $\alpha = \beta = \{1, 2, \ldots, m\}$. We have

$$\det\left(D_j\right) = \prod_{i=j}^{(j+p-1)} \det\left(C_{[[i]]}(\alpha,\alpha)\right) + \sum_{\theta_j,\ldots,\theta_{j+p-2} \neq \alpha} \prod_{i=j}^{(j+p-1)} \det\left(C_{[[i]]}\left(\theta_{i-1},\theta_i\right)\right), \tag{28}$$

where $\theta_{j-1} = \theta_{j+p-1} = \alpha$. The leading principal submatrices $C_i(\alpha, \alpha)$ are bidiagonal with positive entries on their diagonals, so the first term in (28) is positive, and the remaining terms are nonnegative. Hence $\det(D_j) > 0$ and condition (b) is satisfied.

Since D_j is oscillatory it has m distinct, positive eigenvalues $\omega_1, \ldots, \omega_m$, and thus the spectrum of A consists of q zeros together with the numbers

$$\omega_s^{1/p} e^{i2\pi j/p}, \quad j = 1, \ldots, p, \ s = 1, \ldots, m. \tag{29}$$

This completes the proof in the case when b and k are relatively prime.

In the case when $g = \gcd(b, k) > 1$, we first form B as the direct sum of matrices as in (7). The spectrum of A is the union of the spectra of the B_i's, so we apply the previous result to each of the matrices $B_1, \ldots, B_g$. Note that for $i = 1, \ldots, n \bmod g$, B_i is an $(\lfloor n/g \rfloor + 1) \times (\lfloor n/g \rfloor + 1)$ square matrix and for $i = n \bmod g + 1, \ldots, g$, B_i is an $\lfloor n/g \rfloor \times \lfloor n/g \rfloor$ square matrix. The formulas (4) and (5) are obtained by applying the result for relatively prime b and k to each of the matrices B_i and then counting. In the general case, although all of the nonzero eigenvalues are of the form $r\, e^{i2\pi j/p}$, there is no guarantee that the r's are all distinct since it may happen that the spectra of two or more of the B_i's overlap. $\square$

Proof of Corollary 1. Suppose that $a_{b+j,j} > 0$ and $a_{j,k+j} < 0$, and the remaining entries of A are zero. The case when the signs of the diagonals are switched is similar. Let D be the diagonal matrix with diagonal entries

$$d_{jj} = e^{i\pi \frac{j}{b+k}}. \tag{30}$$

Let $B = DAD^{-1}$. Then B has the same band structure as A. Moreover,

$$b_{b+j,j} = e^{i\pi \frac{b}{b+k}} a_{b+j,j}, \quad \text{and} \quad b_{j,k+j} = e^{i\pi \frac{b}{b+k}} e^{-i\pi} a_{j,k+j}. \tag{31}$$

Therefore, A is similar to the following scale multiple of $|A|$:

$$A \sim e^{i\pi \frac{b}{b+k}} |A|. \tag{32}$$

It follows that the eigenvalues of A are the eigenvalues of $|A|$ rotated by $\pi b/(b+k)$, and the corollary is proven. $\square$

Proof of Corollary 2. In the proof of Theorem 1 the form of $C = PAP^T$ in (14) and (25) depended only on the zero pattern of A. So, as long as $a_{ij} = 0$ outside of the two diagonal bands $(b+j, j)$ and $(i, k+i)$, the eigenvalues of A are the pth roots of the eigenvalues of D_j, and hence the spectrum is invariant under rotations by $2\pi/p$. Moreover, if we impose the weaker condition that $a_{b+j,j}, a_{i,k+i} \geqslant 0$, then all minors of the bidiagonal matrices C_i are still nonnegative, so the D_j's in (25) are all totally nonnegative. The eigenvalues of a totally nonnegative matrix are real and nonnegative, but not necessarily distinct and positive [7, cf. Chapter 6, Theorem 2.5]. Thus, we have proved Corollary 2. $\square$

3. An asymptotic problem

As we noted above, the matrix (1) arises in the study of the Generalized Lamé Equation (GLE). Let $\alpha_1, \ldots, \alpha_n$ be distinct complex numbers, and let $\rho_1, \ldots, \rho_n$ be positive numbers. The GLE is the second order ODE given by

$$\prod_{j=1}^{n} (z - \alpha_j)\phi''(z) + 2\sum_{j=1}^{n} \rho_j \prod_{i \neq j} (z - \alpha_i)\phi'(z) = V(z)\phi(z). \tag{33}$$

According to a result of Heine [4], there exist at most $\binom{n+m-2}{m}$ polynomials V of degree $n-2$ for which (33) has a polynomial solution ϕ of degree m. These polynomial solutions are often called *Stieltjes* or *Heine–Stieltjes polynomials*, and the corresponding polynomials V are known as *Van Vleck*

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polynomials. The zeros of the Stieltjes polynomials may be nicely interpreted as the equilibrium positions of particles with unit charge in which particles of charge ρ_j have been fixed at positions α_j in the complex plane and the charges interact according to a logarithmic potential.

Consider the case when the fixed charges all have the same charge ρ and are placed at the vertices of a regular polygon. Eq. (33) is invariant under affine transformations, so in this case the equation may be brought into the following form by moving the fixed charges to the nth roots of unity:

$$\left(z^n - 1\right)\phi''(z) + 2\rho n z^{n-1}\phi'(z) = V(z)\phi(z). \tag{34}$$

We further restrict ourselves to Van Vleck polynomials of the following simple form:

$$V(z) = \mu\left(z^{n-2} - \nu\right). \tag{35}$$

Then obviously the zeros of V are the $(n-2)$th roots of ν. We take this form so that we can relate ν to the eigenvalues of a matrix of the form (1).

Now, letting $\phi(z) = \sum_{j=0}^{m} a_j z^j$ and substituting into (34), we see that μ is determined by

$$\mu = \mu_{m+1} = m(m - 1 + 2\rho n), \tag{36}$$

and ν is the eigenvalue of the matrix A, with corresponding eigenvector $\mathbf{a} = (a_0, a_1, \ldots, a_m)$, where A is the $(m+1) \times (m+1)$ matrix with nonzero entries

$$a_{j+n-2,j} = \frac{\mu - (j-1)(j-2+2\rho n)}{\mu} = 1 - \frac{\mu_j}{\mu_{m+1}}, j = 1, \ldots, m+3-n, \tag{37}$$

$$a_{i,i+2} = \frac{i(i+1)}{\mu}, \quad i = 1, \ldots, m-1. \tag{38}$$

Since μ_j is strictly increasing in j, $a_{j+n-2,j}$ and $a_{i,i+2}$ are all positive. Thus A has the form (1), with $b = n-2$ and $k = 2$. The period of A is $p = n$ if n is odd and $p = n/2$ if n is even. Applying Theorem 1 to A, it immediately follows that the Van Vleck zeros of this form lie on the stars

$$\left\{r\omega^j | r \geqslant 0\right\}, \tag{39}$$

where $\omega = \exp(i2\pi/(n(n-2)/2))$ if n is even, and $\omega = \exp(i2\pi/n(n-2))$ if n is odd.

The question we pose is, what happens when the ρ_j's are not all equal? Borcea and Shapiro [1] conjectured that when $n = 3$, the asymptotic distribution of Van Vleck zeros in the limit as $m \to \infty$ is independent of the ρ_j's. Now, when the ρ_j's are not all equal the Van Vleck zeros are the eigenvalues of a matrix A that is banded, but does not have the double band structure of (1). Rather, there are nonzero entries "inside" the bands. Thus, this conjecture may be reformulated in the following way. Suppose $\{C^{(m)}\}$ is a family of matrices where $C^{(m)}$ is an $m \times m$ matrix of the following form:

$$C^{(m)} = A^{(m)} + B^{(m)}, \tag{40}$$

where $A^{(m)}$ has the double band structure of (1). Suppose that the "perturbation matrices" $B^{(m)}$ are such that their entries are all zero outside of the bands, i.e. $b_{ij}^{(m)} = 0$ for $i - j > b$ or $i - j < -k$. In other words, the double band matrix is perturbed by a banded matrix. Then, what are the conditions on $B^{(m)}$ so that the eigenvalues of $C^{(m)}$ approach the eigenvalues of $A^{(m)}$ in the limit as $m \to \infty$?

Numerical evidence strongly supports the conjecture of Borcea and Shapiro. In this case, however, the perturbations $B^{(m)}$ are highly structured. Nevertheless, standard matrix eigenvalue perturbation theory seems to fail to give a result in this case. We suggest that this conjecture may be a special case of a more general theorem. In particular, we conjecture that it is sufficient for the norm of $B^{(m)}$ to tend to zero as $1/m$.

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