Interlacing and asymptotic properties of Stieltjes polynomials

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Abstract

Polynomial solutions to the generalized Lamé equation, the *Stieltjes polynomials*, and the associated *Van Vleck polynomials* have been studied since the 1830's, beginning with Lamé in his studies of the Laplace equation on an ellipsoid, and in an ever widening variety of applications since. In this paper we show how the zeros of Stieltjes polynomials are distributed and present two new interlacing theorems. We arrange the Stieltjes polynomials according to their Van Vleck zeros and show, firstly, that the zeros of successive Stieltjes polynomials of the same degree interlace, and secondly, that the zeros of certain Stieltjes polynomials of successive degrees interlace. We use these results to deduce new asymptotic properties of Stieltjes and Van Vleck polynomials.

KEYWORDS: Lamé equation, Interlacing zeros, Heine-Stieltjes polynomials, Van Vleck polynomials, Orthogonal polynomials

1 Introduction and main results

Let $\alpha_1 < \cdots < \alpha_p$ be any p distinct real numbers, and let $\rho_1, \ldots, \rho_p$ be positive numbers. The *generalized Lamé equation*[1] is the second order ODE given by

$$S''(x) + \sum_{j=1}^{p} \frac{\rho_j}{x - \alpha_j} S'(x) = \frac{V(x)}{A(x)} S(x) \tag{1}$$

where $A(x) = \prod_{j=1}^{p}(x - \alpha_j)$ and $V(x)$ is a polynomial of degree $p - 2$. A result of Stieltjes [17], known as the Heine-Stieiltjes Theorem [18], says that there exist exactly $\binom{k+p-2}{k}$ polynomials $V(x)$

[1] Usually the term generalized Lamé equation refers to such equations in which the α_j's are allowed to be complex. But here we only consider the real case.

for which (1) has a polynomial solution S of degree k. These polynomial solutions are often called *Stieltjes* or *Heine-Stieltjes polynomials*, and the corresponding polynomials $V(x)$ are known as *Van Vleck polynomials*.

In this paper we will consider the case when $p = 3$, in which case (1) is a Heun equation[2], and the Heine-Stieltjes Theorem says that there are $k + 1$ values of ν for which (1) has a polynomial solution of degree k with $V(x) = \mu(x - \nu)$. We refer to these values of ν as the Van Vleck zeros of order k.

The equation (1) was studied by Lamé in the 1830's in the special case, $\rho_j = 1/2$, $\alpha_1+\alpha_2+\alpha_3 = 0$ in connection with the separation of variables in the Laplace equation using elliptical coordinates [22, Ch. 23]. The equation has since found a strikingly wide variety of other applications, from electrostatics [8, 11, 6, 14] to completely quantum integrable systems such as the quantum C. Neumann oscillators, the asymmetric top and the geodesic flow on an ellipsoid [1, 7, 4, 9]. In particular, the zeros of the Stieltjes polynomials may be nicely interpreted as the equilibrium positions of k unit charges in a logarithmic potential in which at each position α_j in the complex plane is fixed a charge of magnitude ρ_j.

Much is known about the properties of Stieltjes and Van Vleck polynomials for a fixed degree of the Stieltjes polynomial (see, e.g. [21] for recent results), but there are few results relating Van Vleck and Stieltjes zeros of different degrees. A few important facts are that the zeros of any Stieltjes polynomial are simple, lie inside the interval (α_1, α_3), and none of them can equal α_2 or its corresponding Van Vleck zero. Similarly, for fixed k, the Van Vleck zeros are distinct and also lie within (α_1, α_3). The proofs of these results can be found in [20, 18, 16].

Recently we showed in [5] that the Van Vleck zeros of successive orders interlace. That is, if the Van Vleck zeros of order k are written in increasing order as $\nu_1^{(k)} < \nu_2^{(k)} < \cdots < \nu_{k+1}^{(k)}$, then

$$\alpha_1 < \nu_1^{(k+1)} < \nu_1^{(k)} < \nu_2^{(k+1)} < \nu_2^{(k)} < \cdots < \nu_{k+1}^{(k)} < \nu_{k+2}^{(k+1)} < \alpha_3. \tag{2}$$

Much of the research in the past several years has focused on the asymptotic properties of the zeros of Stieltjes and Van Vleck polynomials as the degree of the corresponding Stieltjes polynomials tends toward infinity [3, 2, 13, 12]. Interlacing theorems such as the one above are interesting not only for what they tell us about the zeros for a finite degree, but they also help us to understand such asymptotic limits. They are "classical" results in the sense of being statements about finite degree polynomials, and they are also a bridge to connect other classical results with asymptotic

[2]The Heun equation is (1) with $p = 3$ and where the ρ_j's are allowed to be negative but must satisfy some other conditions (cf. [13]).

limits. Our results below are further steps in this direction.

In this paper we present a theorem on the distribution of the Stieltjes zeros and two additional interlacing theorems. For each positive integer k we label the $k+1$ Stieltjes polynomials of degree k according to their Van Vleck zeros, as $S_j^{(k)}(x)$. That is, $S_j^{(k)}(x)$ is the polynomial of degree k that satisfies

$$\left[\frac{d^2}{dx^2} + \sum_{j=1}^{3} \frac{\rho_j}{x-\alpha_j}\frac{d}{dx} - \frac{\mu\left(x-\nu_j^{(k)}\right)}{A(x)}\right] S_j^{(k)}(x) = 0 \tag{3}$$

Our first result is a strengthening of a classical result of Stieltjes:

Theorem 1. *There are exactly $j-1$ zeros of $S_j^{(k)}$ in the interval (α_1, α_2), and $k-j+1$ zeros of $S_j^{(k)}$ in the interval (α_2, α_3). Moreover, there are no zeros of $S_j^{(k)}$ between α_2 and its corresponding Van Vleck zero $\nu_j^{(k)}$.*

Next we show that the zeros of successive Stieltjes polynomials of the same degree interlace. In the third theorem we establish that the zeros of the jth Stieltjes polynomials of successive degrees interlace. The interlacing properties are illustrated in Figure 1.

Theorem 2. *The zeros of $S_j^{(k)}$ and $S_{j+1}^{(k)}$ interlace; between any two consecutive zeros of $S_j^{(k)}$ there is exactly one zero of $S_{j+1}^{(k)}$. Moreover, the smallest zero of $S_{j+1}^{(k)}$ is less than the smallest zero of $S_j^{(k)}$. That is, let $x_{i,j}^{(k)}$ be the zeros of the Stieltjes polynomial $S_j^{(k)}$, arranged in increasing order. Then*

$$\alpha_1 < x_{1,j+1}^{(k)} < x_{1,j}^{(k)} < x_{2,j+1}^{(k)} < x_{2,j}^{(k)} < \cdots < x_{k,j+1}^{(k)} < x_{k,j}^{(k)} < \alpha_3. \tag{4}$$

Theorem 3. *The zeros of $S_j^{(k)}$ and $S_i^{(k+1)}$ interlace if and only if $i = j$ or $i = j+1$. That is, if $i = j$ or $i = j+1$ then*

$$\alpha_1 < x_{1,i}^{(k+1)} < x_{1,j}^{(k)} < x_{2,i}^{(k+1)} < x_{2,j}^{(k)} < \cdots < x_{k,j}^{(k)} < x_{k+1,i}^{(k+1)} < \alpha_3. \tag{5}$$

Otherwise the zeros of $S_j^{(k)}$ and $S_i^{(k+1)}$ do not interlace.

In the following section we prove these results. Following this, in Section 3, we show how these results can be combined with known asymptotic properties of Stieltjes and Van Vleck polynomials to produce new results. In particular, we construct sequences of Van Vleck zeros to converge to any number in $[\alpha_1, \alpha_3]$, and calculate the asymptotic zero distribution of the Stieltjes polynomials associated with these Van Vleck zeros.

We conclude with a section of remarks and comments on open problems. One of these is whether or not there exist any orthogonal sequences of Stieltjes polynomials. This is a natural

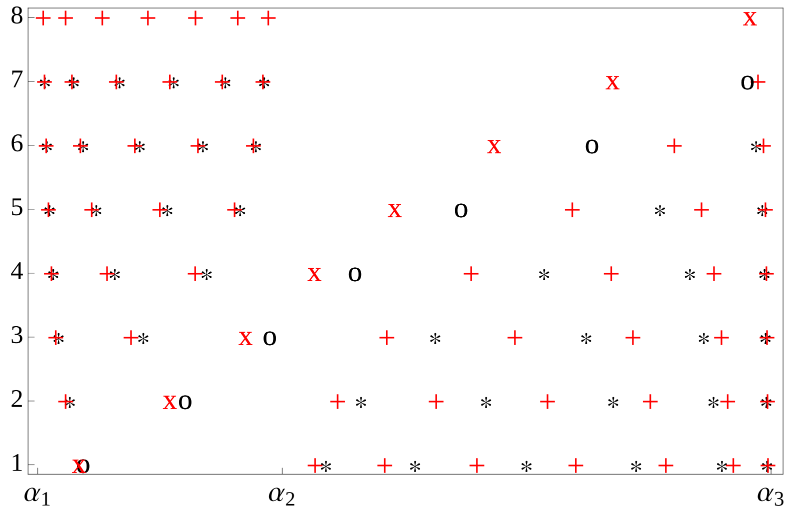

Figure 1: Zeros of Stieltjes and Van Vleck polynomials. On row j are shown $*$: zeros of $S_j^{(k)}$; $+$: zeros of $S_j^{(k+1)}$; O: Van Vleck zero $\nu_j^{(k)}$; X: Van Vleck zero $\nu_j^{(k+1)}$, for $k = 6$. Values of the parameters are $(\alpha_1, \alpha_2, \alpha_3) = (-1, 0, 2)$, $(\rho_1, \rho_2, \rho_3) = (1, 2, 1/3)$.

question to ask, for several reasons. For one thing, we see from Theorem 3 that any sequence of Stieltjes polynomials $\{S_{j_k}^{(k)}\}$ with $j_{k+1} = j_k$ or $j_k + 1$ shares at least one thing in common with orthogonal polynomials, namely the well known fact that the zeros of orthogonal polynomials of successive degrees interlace. Moreover, equation (1) with $p = 2$ is the Jacobi differential equation, and the polynomial solutions are the Jacobi polynomials which are, of course, orthogonal [19]. We will also show that certain of the sequences $\{S_{j_k}^{(k)}\}$ have the same asymptotic zero distribution as sequences of orthogonal polynomials. Additionally, Ismail [10] has shown that under fairly general conditions, equilibrium positions of particles in a logarithmic potential are the zeros of orthogonal polynomials. In our setting, the sufficient condition is only violated by the fact that the term in front of $S'(x)$ in (1) is not differentiable at α_2. Ismail's theorem gives only sufficient conditions for orthogonality, however, so the possibility of orthogonality of Stieltjes polynomials has remained an important open question. It is worth noting that, in spite of these intriguing similarities, in the over 150 year literature on the Lamé equation, we have found no definitive statement or conjecture concerning orthogonality of polynomial solutions of (1) when $p \geq 3$.

2 Proofs of main results

For the proofs of the theorems we will need the following lemma, which is basically the Sturm separation theorem applied in the intervals (α_1, α_2) and (α_2, α_3).

Lemma 1. *There is a zero of $S_{j+1}^{(k)}$ between every two zeros of $S_j^{(k)}$ in the interval (α_1, α_2), and between every zero of $S_j^{(k)}$ in (α_1, α_2) and either of the singular points α_1, α_2. Likewise, there is a zero of $S_j^{(k)}$ between every two zeros of $S_{j+1}^{(k)}$ in the interval (α_2, α_3), and between every zero of $S_{j+1}^{(k)}$ in (α_2, α_3) and either of the singular points α_2, α_3.*

Proof. First we note that the constant μ in (1) is determined by the degree k of the Stieltjes polynomials by substitution and identification of the powers, as

$$\mu = \mu_k = k\,(k - 1 + \rho_1 + \rho_2 + \rho_3)\,. \tag{6}$$

For the remainder of this proof, since we are dealing with Stieltjes polynomials of fixed degree k, we will omit the superscripts and simply write $S_j = S_j^{(k)}$ and $S_{j+1} = S_{j+1}^{(k)}$. Now, the Stieltjes polynomials S_j and S_{j+1} satisfy

$$\left[\frac{d^2}{dx^2} + \sum_{j=1}^{3} \frac{\rho_j}{x - \alpha_j}\frac{d}{dx} - \frac{\mu_k\left(x - \nu_j^{(k)}\right)}{A(x)}\right] S_j(x) = 0, \tag{7}$$

$$\left[\frac{d^2}{dx^2} + \sum_{j=1}^{3} \frac{\rho_j}{x - \alpha_j}\frac{d}{dx} - \frac{\mu_k\left(x - \nu_{j+1}^{(k)}\right)}{A(x)}\right] S_{j+1}(x) = 0. \tag{8}$$

Define the integrating factor

$$J(x) = \prod_{j=1}^{3} |x - \alpha_j|^{\rho_j}\,. \tag{9}$$

Then

$$J'(x) = J(x) \sum_{j=1}^{3} \frac{\rho_j}{x - \alpha_j}. \tag{10}$$

Upon multiplying (7) by S_{j+1} and (8) by S_j, taking the difference of the result, and then multiplying by J, we obtain

$$\frac{d}{dx}\left[J\left(S_{j+1}' S_j - S_{j+1} S_j'\right)\right] = Q S_j S_{j+1}, \tag{11}$$

where

$$Q(x) = (\nu_j - \nu_{j+1})\,\frac{J(x)}{A(x)}. \tag{12}$$

Note that $Q < 0$ in (α_1, α_2) and $Q > 0$ in (α_2, α_3). Now, consider two consecutive zeros of S_j, x_1 and x_2, in the interval (α_1, α_2). Then S_j' must alternate signs at x_1 and x_2. Thus, the expression

$$J\left(S_{j+1}' S_j - S_{j+1} S_j'\right)\Big|_{x=x_1}^{x=x_2} = -\,J\left(S_{j+1} S_j'\right)\Big|_{x=x_1}^{x=x_2} \tag{13}$$

has the same sign as $S_j S_{j+1}$ in (x_1, x_2). But (11) implies that this expression is negative if $S_j S_{j+1} > 0$ in (x_1, x_2) and positive if $S_j S_{j+1} < 0$ in (x_1, x_2). Thus it must be that S_{j+1} changes sign in (x_1, x_2), and we have shown that between any two consecutive zeros of S_j in (α_1, α_2), there is a zero of S_{j+1}.

Now let x_1 be the smallest zero of S_j in (α_1, α_2). Then, since $J(\alpha_1) = 0$,

$$J\left(S'_{j+1}S_j - S_{j+1}S'_j\right)\Big|_{x=\alpha_1}^{x=x_1} = -J(x_1)S_{j+1}(x_1)S'_j(x_1) \tag{14}$$

also has the same sign as S_jS_{j+1} in (α_1, x_1) if S_{j+1} does not change sign in this interval, again contradicting (11). A similar argument can be applied to the largest zero of S_j in (α_1, α_2) and α_2, which establishes that between every zero of S_j in (α_1, α_2) and either of the singular points α_1, α_2, there is a zero of S_{j+1}.

This argument may be exactly repeated in the interval (α_2, α_3), noting that $Q > 0$ in this interval. It follows that between any two consecutive zeros of S_{j+1} in (α_2, α_3), or between α_2 and the smallest zero of S_{j+1} in (α_2, α_3), or between the largest zero of S_{j+1} in (α_2, α_3) and α_3, there is a zero of S_j. □

Proof of Theorem 1. According to a classical result of Stieltjes [17, 22], every possible distribution of the zeros of the Stieltjes polynomials in the intervals (α_1, α_2) and (α_2, α_3) occurs. That is, for each integer k and any integer $0 \le m \le k$, there is a Stieltjes polynomial of degree k with m zeros in (α_1, α_2) and $k - m$ zeros in (α_2, α_3). Thus it suffices to show that there are at least as many zeros of $S_{j+1}^{(k)}$ than of $S_j^{(k)}$ in (α_1, α_2). But this is an immediate consequence of Lemma 1, so the first part of the theorem is proved.

For the second part of the theorem, according to Shah [16, cf. Theorem 2], between any zero of $S(x)$ and ν there is either a zero of $S'(x)$ or α_2. Suppose $\nu > \alpha_2$, and that there is a zero x_1 of $S(x)$ between α_2 and ν. Since there is a zero of $S'(x)$ between x_1 and ν there must be a zero x_2 of $S(x)$ greater than ν. We may assume that x_1 and x_2 are consecutive. But since the zeros of $S(x)$ are simple there cannot be a zero of $S'(x)$ in both intervals (x_1, ν) and (ν, x_2), which is a contradiction. The case when $\nu < \alpha_2$ is similar. □

Proof of Theorem 2. Combining Lemma 1 with Theorem 1, we see that between the $j - 1$ zeros of S_j in (α_1, α_2) there are $j - 2$ zeros of S_{j+1}. The other two zeros of S_{j+1} in (α_1, α_2) lie between α_1 and the smallest zero of S_j and between the largest zero of S_j in (α_1, α_2) and α_2. Similarly, between the $k - j$ zeros of S_{j+1} in (α_2, α_3) there are $k - j - 1$ zeros of S_j. And the other two zeros of S_j in (α_2, α_3) lie between α_2 and the smallest zero of S_{j+1} in (α_2, α_3) and between the largest zero of S_{j+1} in (α_2, α_3) and α_3. We have thus accounted for all of the zeros of S_j and S_{j+1}, and the interlacing is proved. □

We have actually proved a stronger statement than Theorem 2. We note this in the following corollary.

Corollary 1. *Let $l > j$. Then between any two zeros of $S_j^{(k)}$ in (α_1, α_2) there is a zero of $S_l^{(k)}$. Between any two zeros of $S_l^{(k)}$ in (α_2, α_3) there is a zero of $S_j^{(k)}$.*

Proof of Theorem 3. First we prove the "only if" part. In order for the zeros of $S_i^{(k+1)}$ and $S_j^{(k)}$ to interlace, it must be the case that the smallest zero of $S_i^{(k+1)}$ is smaller than the smallest zero of $S_j^{(k)}$, which is impossible if $i < j$ since then there would be fewer zeros of $S_i^{(k+1)}$ than of $S_j^{(k)}$ in the interval (α_1, α_2). Similarly, if $i > j+1$ then there are fewer zeros of $S_i^{(k+1)}$ than of $S_j^{(k)}$ in the interval (α_2, α_3).

For the "if" part, as in the proof of Lemma 1, we derive the following expression

$$\frac{d}{dx}\left[J\left(\frac{dS_i^{(k+1)}}{dx}S_j^{(k)} - S_i^{(k+1)}\frac{dS_j^{(k)}}{dx}\right)\right] = QS_j^{(k)}S_i^{(k+1)}, \tag{15}$$

where in this case

$$Q(x) = (\mu_{k+1} - \mu_k)\frac{J(x)}{A(x)}(x - \hat{\nu}_i), \quad \text{and} \tag{16}$$

$$\hat{\nu}_i = \frac{\mu_{k+1}\nu_i^{(k+1)} - \mu_k\nu_j^{(k)}}{\mu_{k+1} - \mu_k}. \tag{17}$$

Note that (2) implies

$$\hat{\nu}_j < \nu_j^{(k+1)} < \alpha_3 \quad \text{and} \quad \alpha_1 < \nu_{j+1}^{(k+1)} < \hat{\nu}_{j+1}. \tag{18}$$

Suppose, for the moment, that $\alpha_1 < \hat{\nu}_j < \alpha_2$. Then, since $Q > 0$ in $(\hat{\nu}_j, \alpha_2)$, between every two consecutive zeros of $S_j^{(k+1)}$ in $(\hat{\nu}_j, \alpha_2)$, and between the largest zero of $S_j^{(k+1)}$ in $(\hat{\nu}_j, \alpha_2)$ and α_2, there is a zero of $S_j^{(k)}$. Since $Q < 0$ in $(\alpha_1, \hat{\nu}_j)$, there is a zero of $S_j^{(k+1)}$ between every two zeros of $S_j^{(k)}$ in $(\alpha_1, \hat{\nu}_j)$, and between α_1 and the smallest zero of $S_j^{(k)}$ in $(\alpha_1, \hat{\nu}_j)$. This accounts for all of the $j-1$ zeros of $S_j^{(k)}$ and of $S_j^{(k+1)}$ in (α_1, α_2). Similarly, since $Q < 0$ in (α_2, α_3), between every two zeros of $S_j^{(k)}$ in (α_2, α_3) and between every zero of $S_j^{(k)}$ in (α_2, α_3) and either of the singular points α_2, α_3, there is a zero of $S_j^{(k+1)}$. This accounts for the k zeros of $S_j^{(k)}$ and the $k+1$ zeros of $S_j^{(k+1)}$ and the interlacing is proved in this case.

A nearly identical argument holds in the case when $\alpha_2 < \hat{\nu}_{j+1} < \alpha_3$ to show that the zeros of $S_{j+1}^{(k+1)}$ and $S_j^{(k)}$ interlace. The proof is thus completed by the following lemma. □

Lemma 2. $\alpha_1 < \hat{\nu}_j < \alpha_2$ *and* $\alpha_2 < \hat{\nu}_{j+1} < \alpha_3$.

Proof. Suppose that $\hat{\nu}_j \leq \alpha_1$. Then it would be the case that $Q > 0$ in (α_1, α_2) and by arguments similar to the above, between every two zeros of $S_j^{(k+1)}$ in (α_1, α_2) and between every zero of $S_j^{(k+1)}$ in (α_1, α_2) and either of the singular points α_1, α_2, there is a zero of $S_j^{(k)}$. But this would imply the existence of at least j zeros of $S_j^{(k)}$ in (α_1, α_2), which contradicts Theorem 1. Likewise, if $\hat{\nu}_j \geq \alpha_2$, an similar argument would imply the existence of at least j zeros of $S_j^{(k+1)}$ in (α_1, α_2), which is also a contradiction. This argument may be repeated to prove the second statement. □

3 Asmyptotic properties

We now investigate how the zeros of Stieltjes polynomials distribute over the interval (α_1, α_3) in the limit as the degree of the Stieltjes polynomials tends to infinity. Recall that Theorem 1 says that if $\nu < \alpha_2$, then there are $j-1$ zeros of $S_j^{(k)}$ in (α_1, ν) and $k-j+1$ zeros in (α_2, α_3), and similarly if $\nu \geq \alpha_2$, *mutatis mutandi.*

In [3] it is shown that the Van Vleck zeros have the asymptotic distribution given by a probability measure supported on (α_1, α_3), with density $\rho_V(x)$ given by the following:

$$\rho_V(x) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{\alpha_2}^{\alpha_3} \frac{ds}{\sqrt{A(s)(x-s)}} & \text{if} \quad \alpha_1 < x < \alpha_2, \\ \dfrac{1}{2\pi} \displaystyle\int_{\alpha_1}^{\alpha_2} \frac{ds}{\sqrt{A(s)(x-s)}} & \text{if} \quad \alpha_2 < x < \alpha_3. \end{cases} \tag{19}$$

Thus, given any point $\nu \in (\alpha_1, \alpha_3)$ there is a sequence of Van Vleck zeros $\left\{\nu_{j_k}^{(k)}\right\}$ that converges to ν as $k \to \infty$. So we may ask, What is the distribution of the corresponding Stieltjes zeros in this same limit? A partial answer to this question is given by Theorem 1, which implies that the limiting density is supported on a subset of $(\alpha_1, \alpha_3) \setminus I$, where I is the interval bounded by α_2 and ν. The question can be resolved by applying the results of [13].

Martínez-Finkelshtein and Saff [13] consider the more general case of (1) for an arbitrary number p of α_i's. The authors fix relative proportions $\theta_1, \ldots, \theta_{p-1}$ in each of the intervals (α_i, α_{i+1}), $i = 1, \ldots, p-1$, and extract a sequence of Stieltjes polynomials such that as the degree of each polynomial in the sequence tends to infinity, the fraction of its zeros in the interval (α_i, α_{i+1}) tends to θ_i. Under this assumption the authors derive asymptotic results for the zeros of Stieltjes and Van Vleck polynomials. We specialize as before to the $p = 3$ case and recast these calculations in the light of our results.

Let $0 \leq \theta \leq 1$. Theorem 1 of [13] gives the asymptotic distribution of the zeros of a sequence of Stieltjes polynomials. We interpret these in our setting in the following way. Suppose the sequence

$\left\{S^{(k)}_{j_k}\right\}$ is chosen such that the fraction of the zeros of $S^{(k)}_{j_k}$ in (α_1, α_2) tends to θ. Some simple but tedious calculations shows that the limit $\lim_{k\to\infty} \nu^{(k)}_{j_k} = \nu$ of the corresponding Van Vleck zeros is determined by

$$\frac{1}{\pi}\int_{\alpha_1}^{\min(\alpha_2,\nu)} \sqrt{\left|\frac{\nu - x}{A(x)}\right|}dx = \theta. \tag{20}$$

Moreover, let I by the interval bounded by α_2 and ν. Then the asymptotic distribution of the Stieltjes polynomials $S^{(k)}_{j_k}$ is given by

$$\rho_S(x) = \begin{cases} \dfrac{1}{\pi}\sqrt{\dfrac{\nu - x}{A(x)}} & \text{if} \quad x \in (\alpha_1, \alpha_3) \setminus I \\ 0 & \text{if} \quad x \in I \end{cases} \tag{21}$$

We note, in particular, that if $\nu = \alpha_1, \alpha_2$ or α_3 then ρ_S is the so-called "arcsine distribution" supported on the intervals (α_2, α_3), (α_1, α_3) and (α_1, α_2), respectively.

Now, by Theorem 1, $j_k - 1$ zeros of $S^{(k)}_{j_k}$ lie in (α_1, α_2). So, if $j_k/k \to \theta$, then the fraction of zeros of $S^{(k)}_{j_k}$ in (α_1, α_2) tends to θ. (One such sequence, that also satisfies $j_{k+1} = j_k$ or $j_k + 1$, is $j_k = \lceil k\theta + 1\rceil$.) Combining our results with the asymptotic properties, we have the following.

Theorem 4. *(i) Let $\{j_k\}$ be a sequence of positive integers such that $j_k/k \to \theta$. Then we have the limit of sequences of Van Vleck zeros:*

$$\lim_{k\to\infty} \nu^{(k)}_{j_k} = \nu, \tag{22}$$

where ν is determined by (20). *(ii) Given any $\nu \in [\alpha_1, \alpha_3]$, if θ is defined by* (20) *and $j_k/k \to \theta$, then the sequence $\left\{\nu^{(k)}_{j_k}\right\}$ converges to ν. (iii) If the sequence $\{j_k\}$ satisfies $j_{k+1} = j_k$ or $j_k + 1$, then the sequence of Stieltjes polynomials $\left\{S^{(k)}_{j_k}\right\}$ is an infinite sequence of polynomials with interlacing zeros and asymptotic zero distribution given by* (21).

We may also calculate conditions under which the limiting distribution of Stieltjes zeros is supported throughout (α_1, α_3). A necessary condition given by Theorem 1 is that $\nu = \alpha_2$, and (21) tells us that this is also a sufficient condition. In order for the limiting distribution to be supported on (α_1, α_3), the fraction of the zeros of $S^{(k)}_{j_k}$ in (α_1, α_2) must tend to θ_c, where θ_c is calculated by setting $\nu = \alpha_2$ in (20). A simple calculation shows that this is (see also [13, Prop. 1]):

$$\theta_c = \frac{2}{\pi}\sin^{-1}\sqrt{\frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_1}}. \tag{23}$$

4 Concluding remarks and open questions

It is interesting that the asymptotic properties of Stieltjes and Van Vleck polynomials are independent of the fixed charges ρ_j, and depend only on the locations α_j of the fixed charges.

Thus, for any configuration of the α_j's, there is a three parameter (ρ_1, ρ_2, ρ_3) family of Stieltjes polynomials with the same asymptotic properties. Of course, for finite k the zeros of $S_j^{(k)}$ depend on the ρ_j's, and so our results are a way to connect the classical results on Stieltjes and Van Vleck polynomials to the more recent work done on the asymptotic properties of these functions. This allows us to present a fairly complete description in the case we have considered of real α_j's and $p = 3$.

An obvious way to generalize the results we have presented would be to consider the case of more general p. In this case the Van Vleck polynomials are of degree $p - 2$. Hence one question that arises is how to order the Van Vleck and Stieltjes polynomials in a manner analogous to the ordering defined in (3). It may be that there is no generally consistent ordering as there is for $p = 3$. However, it still may be possible to generalize some of the results of this paper.

Perhaps the most important open question in the $p = 3$ case is whether there exist sequences of orthogonal Stieltjes polynomials. Orthogonality is a rather strict condition on sequences of polynomials, but, as we mentioned in the Introduction, there are compelling reasons to suspect that some sequences of Stieltjes polynomials might be orthogonal. It is well known (see, e.g. [18]) that if a sequence $\{p_n\}$ of monic polynomials of degree n is orthogonal with respect to some measure, then the zeros of p_n and p_{n+1} interlace. Moreover, $\{p_n\}$ is orthogonal if and only if there exist sequences $\{a_n\}$ and $\{b_n\}$, with $a_n \in \mathbb{R}$ and $b_n > 0$ such that $p_n = (x - a_n)p_{n-1} - b_n p_{n-2}$. We have found no simple contradiction to emerge from assuming that this holds for sequences of Stieltjes polynomials.

Additionally, if $a_n \to a \in \mathbb{R}$ and $b_n \to b \in (0, \infty)$, then the zeros of p_n are described asymptotically by an arcsine distribution [15, cf. Theorem 5.3]. This leads us to another property shared by orthogonal polynomials and certain sequences of Stieltjes polynomials: When ν in (21) is one of $\alpha_1, \alpha_2, \alpha_3$, or equivalently, if $\theta = 0, 1$ or θ_c, the sequence $\left\{ S^{(k)}_{\lceil k\theta + 1 \rceil} \right\}$ is an infinite sequence of polynomials with interlacing zeros and asymptotic zero distribution supported in an open interval and identical to the asymptotic zero distribution of a sequence of orthogonal polynomials. These sequences thus share many properties of orthogonal polynomial sequences. However, despite these commonalities, we think it is unlikely for sequences of Stieltjes polynomials to be orthogonal.

We conclude by noting that an important conjecture in quantum chaos due to M. V. Berry and M. Tabor asserts that for generic quantum integrable systems, the mean level spacing should exhibit a random distribution in the semi-classical limit. It is also believed that a similar behavior holds for the zeros of the corresponding eigenfunctions. An interesting application of our interlacing results may consist of computing the mean level spacing of the quantum integrable systems mentioned in the introduction, and see if the Berry-Tabor conjecture holds in these cases.

References

[1] A. Agnew and A. Bourget. Semi-classical density of states for the quantum asymmetric top. *J. Phys. A: Math. Theor.*, 41(18), 2008.

[2] J. Borcea. Choquet order for spectra of higher Lamé operators and orthogonal polynomials. *J. Approx. Theory*, 151(2):164–180, 2008.

[3] J. Borcea and B. Shapiro. Root asymptotics of spectral polynomials for the Lamé operator. *Comm. Math. Phys.*, 282(2):323–337, 2008.

[4] A. Bourget and T. McMillen. Spectral inequalities for the quantum asymmetrical top. *J. Phys. A: Math. Theor.*, 42(9), 2009.

[5] A. Bourget, T. McMillen, and A. Vargas. Interlacing and non-orthogonality of spectral polynomials for the Lamé operator. *Proc. Amer. Math. Soc.*, 137(5):1699–1710, 2009.

[6] D.K. Dimitrov and W.V. Assche. Lamé differential equations and electrostatics. *Proc. Amer. Math. Soc.*, 128(12):3621–3628, 2000.

[7] M.P. Grosset and A.P. Veselov. Lamé equation, quantum top and elliptic Bernoulli polynomials. *arXiv:math-ph/0508068v2*.

[8] F. A. Grünbaum. Variations on a theme of Heine and Stieltjes: an electrostatic interpretation of the zeros of certain polynomials. *J. Comput. Appl. Math.*, 99(1-2):189–194, 1998.

[9] J. Harnad and P. Winternitz. Harmonics on hyperspheres, separation of variables and the Bethe ansatz. *Lett. Math. Phys.*, 33:61–74, 1995.

[10] M. E. H. Ismail. *Classical and orthogonal polynomials in one variable*. Cambridge Univ. Press, 2005.

[11] F. Marcellán, A. Martínez-Finkelshtein, and P. Martínez-González. Electrostatic models for zeros of polynomials: old, new, and some open problems. *J. Comput. Appl. Math.*, 207(2):258–272, 2007.

[12] A. Martínez-Finkelshtein, P. Martínez-González, and R. Orive. Asymptotics of polynomial solutions of a class of generalized Lamé differential equations. *Electron. Trans. Numer. Anal.*, 19:18–28, 2005.

[13] A. Martínez-Finkelshtein and E. B. Saff. Asymptotic properties of Heine-Stieltjes and Van Vleck polynomials. *J. Approx. Theory*, 118(1):131–151, 2002.

[14] T. McMillen, A. Bourget, and A. Agnew. On the zeros of complex Van Vleck polynomials. *J. Comput. App. Math.*, 223(2):862–871, 2009.

[15] P. G. Nevai. Orthogonal polynomials. *Mem. Amer. Math. Soc.*, 18(213):v+185, 1979.

[16] G. M. Shah. On the zeros of Van Vleck polynomials. *Proc. Amer. Math. Soc.*, 19:1421–1426, 1968.

[17] T. J. Stieltjes. Sur certains polynômes qui vérifient une équation différentielle linéaire du second ordre et sur la theorie des fonctions de Lamé. *Acta Math.*, 6(1):321–326, 1885.

[18] G. Szegö. *Orthogonal polynomials.* American Mathematical Society, Providence, R.I., fourth edition, 1975.

[19] V. Totik. Orthogonal polynomials. *Surveys in Approximation Theory*, 1:70, 2005.

[20] E.B. Van Vleck. On the polynomials of Stieltjes. *Bull. Amer. Math. Soc.*, 4:426–438, 1898.

[21] H. Volkmer. Four remarks on eigenvalues of the Lamé equation. *Analysis and Applic.*, 2:161–175, 2004.

[22] E. T. Whittaker and G. N. Watson. *A course of modern analysis.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, Reprint of the fourth (1927) edition, 1996.