On the falling (or not) of the folded inextensible string

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Abstract

This paper concerns the behavior of the folded inextensible string. We show that the unique solution to the idealized folded string problem is the static solution; a string sharply folded onto itself is in an unstable equilibrium. Previous studies of this problem have reached contradictory conclusions: one treatment claims the free end of the idealized folded string falls with a speed that approaches infinity, another claims the speed of the free end falls like an object in free fall, and yet another claims an infinity of solutions. We will demonstrate the flaws in the previous treatments, and show that they result from a failure to consider both the equations of motion and the constitutive relation for an inextensible string. Results on the falling rod are included to demonstrate the effects of stiffness. As the stiffness of the rod tends to zero, the behavior of the rod tends to that of the string, but not in a monotonic manner.

1 Introduction

Hold two ends of a chain and release one end at the same time that a ball is released from the same height. Somewhat counterintuitively, the chain wins the race.

One purpose of this paper is to clarify some contradictory and incorrect results in previous studies of the falling string and falling rod. The falling string (Fig. 1) is treated in several textbooks on theoretical mechanics ([6], for example). The usual treatment (referred to in this paper as the "traditional analysis") is to approach it as an energy problem, assume that the total energy is conserved, and derive the speed of the falling part of the string, without considering the governing equations of motion. It is argued that since the mass of the section of the string that is falling continually decreases, energy conservation requires that the speed of the falling section approach infinity as the end of the string reaches the fold.

However, the problem is not so simple as it at first seems, and conflicting results have been obtained. The claim is made in [9] that the traditional analysis is incorrect; rather, it is argued that the end falls with an acceleration of g, as in free fall. The justification for this result lies in a supposed error of the previous treatments. The authors claim that there must be a continuous loss of kinetic energy as the element on the falling side passes the fold. An *ad hoc* explanation for the loss of kinetic energy is given: there is a continuous series of "plastic impacts" at the fold as elements pass from the falling side to the static side of the fold. However, this explanation is not justified by any mechanical principle, and amounts to the assumption that the part in motion falls with acceleration g. An additional term accounting for plastic impacts, which has no justification in the mechanics of elastic strings, is added to the Lagrange equations in order to achieve the result.

This error is repeated in a study of a closely related problem, the falling chain [8]. The authors take a chain with ends held a distance apart, drop one end and observe with high-speed photography

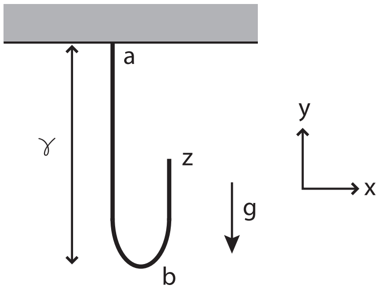


Figure 1: The falling folded string, as pictured in the traditional analysis. The string is attached at point a, which is fixed. The end of the string is labelled z; the fold is labelled b; d is the distance from the fixed point a to the fold b. The force of gravity is in the $-y$ direction.

that the end of the chain falls faster than a ball. The experimental study was supplemented with an analysis of the equations of motion of the chain, and numerical solutions matched the experimental result. Furthermore, the authors show that the equations for the chain are identical to the discretized equations for the string, if a proper discretization is used. Moreover, they provided an explanation for the observed phenomenon: as the infinitesimal element on the falling side of the fold passes the fold, the kinetic energy of this element is transferred to the moving part of the string, causing it to fall faster than an object under free fall. Thus, they showed that as the initial configuration of the string approaches an idealized fold, that the speed of the falling end approaches that obtained through the traditional analysis. This result contradicted the prediction made in [9], however, that the end of the falling folded string should fall like a free falling object, and some confusion remains, as it is stated in the conclusion of [8] that the solution obtained is not unique, that another solution exists with the plastic impacts asserted in [9]. An additional paper [7] contradicts the earlier paper [9] by claiming an infinity of solutions, including the result in [9] that the folded string falls with an acceleration g. In a further paper [5] a solution is derived that contains an arbitrary parameter which contains the infinity of solutions in [7].

Thus far, no explanation has been given for why the prediction in [9] turned out to be false. Nor has there been an explanation for the contradiction between the claims of an unique solution in [9] and an infinity of solutions in [7]. We will attempt to clarify this matter, and show that the traditional analysis is correct, but only in a limiting sense. We will show that the unique solution to the folded string equations is, in fact, the static solution: the idealized folded string doesn't move at all. It is well known that for a smooth configuration of the string, the energy of the string is conserved. Thus, the confusion results from a lack of clarity over what occurs in the discontinuous case, when the string is sharply folded onto itself. We will derive rational equations for the string, the solution for the sharply folded string, and show how the traditional result arises in a limit as the initial condition approaches the ideal folded string.

We will also consider falling rods to determine the effect of stiffness on the falling speed. The inextensible string may be viewed as the limit of an inextensible rod as bending stiffness tends to zero. One question is how does increasing stiffness affect the speed of the falling side. This problem is briefly addressed in [3]. Numerical studies of a beam model show that the speed of a falling folded beam increases as the stiffness decreases. But, it is unclear why the beam model should approach the string model, and only two values of the stiffness are considered. In fact, the result is not, in general, correct. An increase in stiffness can also increase the speed of the falling end. This is clear if we consider a rod with high stiffness. Fold the rod and then let one end fall. The bending stiffness will cause the rod to tend to straighten out, causing the end to fall faster than a rod with low stiffness. That is, the rod is not only falling, but “springing” downward, as the elastic energy of the rod is transferred into its falling. The result that a decrease in stiffness results in an increase in the speed of falling is true only for low values of the stiffness, when the damping effect of stiffness overcomes its “springiness.”

2 The string and the chain

We begin by examining how the idealized folded string falls under the force of gravity. A *string* is a *filament*–a long, thin elastic structure–with no bending stiffness. So, consider the configuration in Fig. 1. We will approach this in two ways. First we will present the traditional analysis, without deriving the equations of motion. Then we will consider the equations of motion for the string and see how the traditional analysis is flawed, but represents a limiting case of the folded string. Figure 1 represents a folded string, that is, the fold at point b is supposed to be infinitesimal. The figure is slightly misleading in this respect, for it does not express the fact that there is a discontinuity in derivatives at the fold. The angle between the tangent to the curve and the x-axis makes a 180-degree jump at the fold, and the falling part of the string is actually in the same space as the piece on the “left” side of the fold. This is therefore an unrealistic idealization. Figure 2 is a more accurate representation of the folded string.

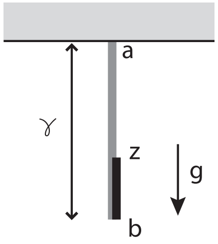


Figure 2: The folded string–an accurate representation. The grey section is $s = 0$ to $s = \gamma$. The black is from $s = \gamma$ to $s = l$. The height of the end of the string is $z = l - 2\gamma$, measured from a; thus $z = 0$ initially, when $\gamma = l/2$ and the free end is at a.

2.1 The traditional analysis

We seek an equation of motion for the point z, the height of the falling end of the string. The falling end of the string falls at twice the speed that the fold moves, i.e. $\dot{z} = -2\dot{\gamma}$, where the superposed dot denotes differentiation with respect to t. The energy of the string is the sum of kinetic and potential energies. With a proper scaling we set the mass per unit length of the string equal to 1. Setting the potential at a equal to zero, the energy is,

$$E = 2\dot{\gamma}^2(l-\gamma) + \frac{1}{2}g\left[l^2 - 4l\gamma + 2\gamma^2\right], \tag{1}$$

The derivative of the energy is

$$\frac{dE}{dt} = -\left[2\left(g - 2\ddot{\gamma}\right)(l-\gamma) - 2\dot{\gamma}^2\right]\dot{\gamma}. \tag{2}$$

If energy is conserved and $\dot{\gamma} \neq 0$, then we find the following equation for either γ or z:

$$\ddot{\gamma} = \frac{1}{4}g\left[1 + \frac{l^2}{4(l-\gamma)^2}\right], \quad \text{or} \quad \ddot{z} = -\frac{1}{2}g\left[1 + \frac{l^2}{(l+z)^2}\right]. \tag{3}$$

According to this equation, at $t = 0$, $\ddot{z} = -g$, and $\ddot{z}$ decreases monotonically as z approaches the fold, becoming unbounded as it reaches the fold. Equation (3) can be solved in terms of elliptic functions (Fig. 3). In particular, the time for the end of the string to fall the length of the string is

$$\sqrt{\frac{l}{g}} \int_0^1 \sqrt{\frac{u}{1-u^2}} du \approx 1.198 \sqrt{\frac{l}{g}}, \tag{4}$$

compared to a falling time of $\sqrt{2l/g} \approx 1.414\sqrt{l/g}$ for a free falling object falling a length l. The ratio of the falling times is therefore a constant (≈ 0.847) that does not depend on the length of the string.

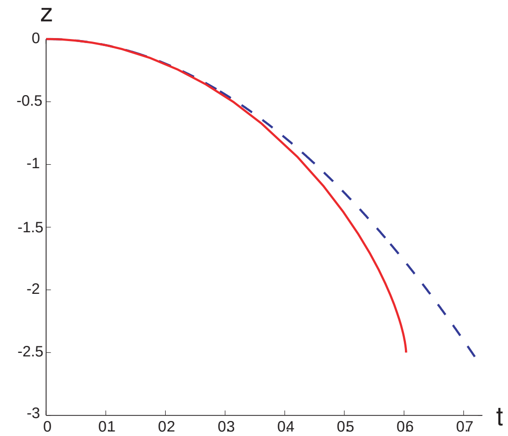


Figure 3: The end of the falling folded string, 2.5 m long, according to equation (3). The dashed curve is the position of an object falling under gravity. Note that $\dot{z} \to -\infty$ as $z \to -l$.

Equation (3) governs the behavior of the falling end of the string, then, under the following conditions: (i) the energy of the string is given by (1), (ii) the energy is conserved, and (iii) the string is moving: $\dot{\gamma} \neq 0$. Each of these conditions must be justified, which can only be done by considering the equations of motion of the string, which we now do. (See [1] for a full derivation.)

2.2 The string equations and the static solution

We consider an inextensible elastic string. A *configuration* of a string is given at each time t by a space curve $s \mapsto \mathbf{r}(s,t)$, where $s \in [0,l]$ is the arc-length of the curve. A string is a perfectly flexible elastic rod. The requirement of perfect flexibility is expressed by

$$\mathbf{n} \times \mathbf{r}_s = 0, \tag{5}$$

where $\mathbf{n}(s,t)$ is the *contact force* exerted by $[0,s)$ on $(s, s+dx)$. In other words, the contact force is (almost) everywhere tangent to the curve $\mathbf{r}(s,t)$. Equation (5) can be taken as the definition of a string, and is equivalent to the requirement that there exists a scalar-valued function $N(s,t)$, called the *tension*, such that

$$\mathbf{n} = N \frac{\mathbf{r}_s}{|\mathbf{r}_s|}. \tag{6}$$

We consider a string confined to the plane, with $\mathbf{n}(s,t) = F\,\mathbf{i} + G\,\mathbf{j}$, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, with a gravity field g along the y-direction. (In [8] it is shown that the equations governing a chain with n links are the same as those that result from the discretization of the string equations into n space steps.

A string can be considered as a chain with infinitely many links; a chain confined to the plane is a chain either constrained by a guidance, or one that allows bending in the normal direction only, such as a bicycle chain.) For a smooth configuration, the balance of linear momentum gives (cf [1], p.15)

$$\rho A\,\ddot{\mathbf{r}} = \mathbf{n}_s + \rho A\,\mathbf{f}, \tag{7}$$

where ρA is the mass per unit length, which we again set equal to one with a proper scaling, and $\mathbf{f}$ is the body force. The force of gravity is in the $-y$ direction, so $\mathbf{f} = -g\,\mathbf{j}$, where g is positive. However, we are considering a case where solutions are not continuous. A weak formulation of the solutions (or, equivalently, invoking the principle of virtual power) in the discontinuous case yields (7) for the sets on which the configuration is smooth. For the discontinuity at the fold, the force and speed must obey a Rankine-Hugoniot jump condition (cf [1], p.28):

$$[[\mathbf{n}]] + \dot{\sigma}[[\dot{\mathbf{r}}]] = 0, \tag{8}$$

where $[[\mathbf{n}]] = \mathbf{n}(\gamma^+, t) - \mathbf{n}(\gamma^-, t)$ is the jump across the discontinuity at γ, and $\dot{\sigma}$ is the speed of the discontinuity.

In order to close the system we need a *constitutive relation*, a relation between the stress and the strain. We are considering a string that is *inextensible*, that is, it does not elongate under tension or shrink under compression. This is expressed by

$$|\mathbf{r}_s| = 1. \tag{9}$$

The constitutive relation (9), along with (6, 7), forms a closed system of equations. We define the quantities

$$T = \frac{1}{2}\dot{\mathbf{r}}^2 - \mathbf{f}\cdot\mathbf{r}, \quad X = \mathbf{n}\cdot\dot{\mathbf{r}}. \tag{10}$$

The inextensibility condition (9) and the defining property of the string (6) imply $\mathbf{n}\cdot\dot{\mathbf{r}}_s = 0$. And, since $\dot{\mathbf{f}} = 0$, the following conservation law holds where the derivatives exist:

$$T_t = X_s. \tag{11}$$

We define the energy of the string,

$$\mathcal{H}(t) \overset{\text{def}}{=} \int_0^l T\,ds, \tag{12}$$

so that

$$\frac{d\mathcal{H}}{dt} = \int_0^l X_s\,ds. \tag{13}$$

Energy is conserved if and only if the integral on the RHS of (13) vanishes, that is, iff X is differentiable and vanishes at the endpoints. But, energy is not, in general, conserved for discontinuous solutions such as the case we are considering. Despite the fact that we do not know *a priori* whether or not the energy is conserved, the equation (13) provides the key to the falling folded string problem.

Now we consider the configuration of the falling string as represented in Fig. 2. In coordinates this is,

$$x = 0 \tag{14}$$

$$y = \begin{cases} -s & \text{if } 0 < s < \gamma(t) \\ s - 2\gamma(t) & \text{if } \gamma(t) < s < l \end{cases} \tag{15}$$

In order to derive an equation for $\gamma(t)$, we will utilize the equations of motion and the constitutive relation. First, we determine the contact force $\mathbf{n}$. The boundary conditions for the force for a string fixed at $s = 0$ and free at $s = l$ are

$$\partial_s F(0,t) = 0, \;\; \partial_s G(0,t) = g, \;\; F(l,t) = G(l,t) = 0. \tag{16}$$

Thus, if (14-15) is a solution of (7) then $F = 0$ and

$$G = \begin{cases} g(s-l) - 2\ddot{\gamma}(\gamma - l) - 2\dot{\gamma}^2 & \text{if } 0 \leq s < \gamma(t), \\ (g - 2\ddot{\gamma})(s - l) & \text{if } \gamma(t) < s \leq l, \end{cases} \tag{17}$$

A simple calculation shows that (14-15, 17) satisfies the jump condition (21), so we need the global equation (13) to determine an equation for γ. Substituting (14-15) into (12), we get an expression for the energy $\mathcal{H}$ that is identical to that in (1), i.e.

$$\frac{d\mathcal{H}}{dt} = 2\dot{\gamma}\,(2\ddot{\gamma} - g)\,(l - \gamma) - \dot{\gamma}^3. \tag{18}$$

Setting the derivative of $\mathcal{H}$ equal to zero yields the traditional result (3). However, the RHS of (13) is

$$2\dot{\gamma}\,(2\ddot{\gamma} - g)\,(l - \gamma). \tag{19}$$

Therefore, equation (13) implies

$$\dot{\gamma} = 0. \tag{20}$$

We see that an inextensible string in the configuration of Fig. 2 is in equilibrium. We emphasize that this is a *unique* solution that follows directly from the three conditions of perfect flexibility (5), the balance of linear momentum (7), and inextensibility (9). This is an unstable static solution analogous to the pendulum equilibrium with the shaft of the pendulum straight above the pivot point. This is also a physically unrealistic solution, since the string is sharply folded onto itself. In terms of a chain with n links, the solution (14-15) represents the situation where the first k links are hanging straight down, and the links from k to n are poised vertically above their pivots, exactly like the unstable pendulum solution, but with $n - k$ attached pendula.

2.3 Critique of previous treatments

Previous analyses have relied on balancing the linear momenta and deriving expressions for the jump in tension across the fold. The tension at the fold is determined by noting that the y component of the contact force is $G = Ny_s$. Then a simple calculation shows that the jump condition (8) can be written in terms of the tension N as,

$$N(\gamma^-, t) + N(\gamma^+, t) = 2\dot{\gamma}^2. \tag{21}$$

For the static solution $\dot{\gamma} = 0$, the tension is discontinuous $N(\gamma^-, t) = -N(\gamma^+, t) = g(l - \gamma)$, due to the fact that for $s < \gamma$ the string is under tension and for $s > \gamma$ it is under compression. (See Fig. 6.)

Steiner and Troger [9] argue that since infinitesimal elements on either side of the fold (Fig. 4) have the same potential energy, but the element on the falling side has a positive kinetic energy, there must be a continuous loss of energy. Under this condition, balancing the momentum across the fold requires $N(\gamma^+, t) = 0$, $N(\gamma^-, t) = 2\dot{\gamma}^2$, which satisfies the jump condition (21). This leads the authors to the equation $2\ddot{\gamma} = g$, as for an object in free fall. The flaw in this reasoning is in assuming that the string in the configuration of Fig. 2 *will* fall and will remain in that configuration as it falls. But, as we have seen, a string in that configuration is in equilibrium, so $2\ddot{\gamma} = g$ violates the governing equations (7). If the string is falling, then it cannot be in the configuration of Fig. 2, and hence the assumption that an infinitessimal element on the left side of the fold does not move is not warranted. Furthermore, the authors assume that the loss of energy in the element dx as it passes from one side of the fold to the other cannot be transfered to the moving portion of the string, which is also unwarranted. The equations in [9] relating the values of speed and tension across the fold amount to the equation $0 = 0$, which is why the result that the free end falls like a free-falling object did not immediately appear as a contradiction.

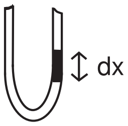


Figure 4: Infinitesimal element dx near the fold.

Schagerl [7] posits an infinity of solutions to the folded string problem by considering the equations of motion (7) and the jump condition at the fold (21). By claiming that the jump remains unspecified, it is claimed that there are an infinity of solutions obtained by taking *all* values of $N(\gamma^{\pm}, t)$ that satisfy (21). But, again, this implies that there are solutions of the string equations in which the string in the configuration in Fig. 2 moves, which, as we have seen, is not the case. Furthermore, Steiner and Troger [9] posited a *unique* solution. The infinity of solutions in [7] includes that in [9], but offers no explanation for the discrepancy between the fact that previous authors found a unique solution, whereas an infinity of solutions are then proposed. O'Reilly and Varadi [5] repeat this mistake in their study of shocks. Based on an analysis of the jump condition, they posit a solution that satisfies the Clausius-Duhem inequality, but there appears in their solution an arbitrary parameter. This arbitrary parameter appears because they do not consider the equation for the energy (13), and thus leads to a recapitulation of the error of Schagerl.

The flaws in these treatments stem from the failure to consider *global* equations like (13) that relate quantities depending on the entire length of the string. As we have seen, this equation is a result of the inextensibility condition, which gives us the RHS of (13). Without taking proper account of the constitutive relation, the rate of change of the energy is not determined, which can lead to incorrect results.

2.4 Numerical results for the falling string

Since the idealized folded string doesn't move at all, in order to consider a falling string, we must consider a string with a finite, smooth fold. A string with a smooth configuration, fixed at one end and free at the other obeys energy conservation due to equation (13). Thus, we expect that for smooth configurations that approach the idealized folded string as in Fig. 2, the speed of the falling end should approach that obtained by the traditional analysis. This is precisely what happens. Previous authors have approached this in different ways. Schagerl [7] considers a string constrained by a guidance. Steiner and Troger [8] solve the chain equations with an initial condition in a box shape, with $n-1$ links hanging straight down and one horizontal link connecting them at the fold. In both cases, (3) is recovered as a limit. Equation (3) represents the behavior of the falling string in the limit as the size of the fold goes to zero. As we have seen, though, it does not represent the actual limit of the idealized folded string. Again, the analogy is with the simple pendulum. If the initial condition of a simple pendulum is taken closer and closer to the straight position above the pivot, the maximal speed of the pendulum as it traverses its orbit increases (although not to infinity). But, if the initial condition actually *is* the straight position above the pivot, it doesn't move at all.

We consider the case of holding the ends of a string some distance Δ apart, and releasing one end. This is not only a more realistic case than those considered previously, but it reveals some interesting behavior. Before the end is released the equilibrium configuration is a catenary (Fig. 5). Setting

$$\delta = \frac{\Delta}{l}, \tag{22}$$

the initial catenary is a horizontal line for $\delta = 1$, and the folded string of Fig. 2 for $\delta = 0$.

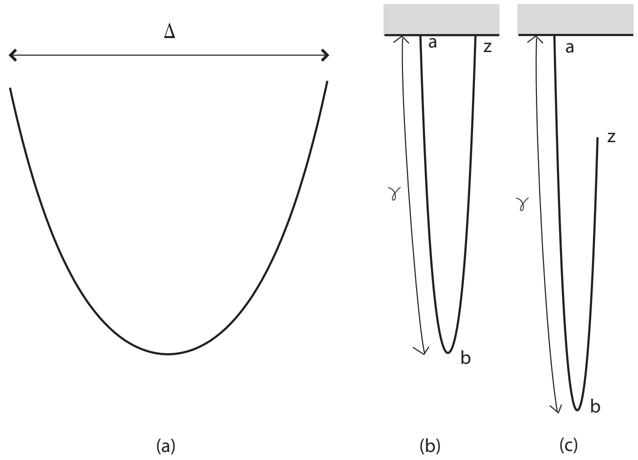


Figure 5: (a) The catenary for $\delta = 0.5$. (b) Initial condition at $t = 0$ for $\delta = 0.1$, and (c) configuration after a short time $t > 0$.

In order to study the behavior of the falling string under various initial conditions, we turn to numerical solutions of the equations. The discretized string equations, with n space steps, are equivalent to the equations of motion for a chain with n links. We find that if δ is small then the string is described by the solution (14-15,17), with corrections of order δ. As $\delta \to 0$ the behavior is governed by (3), and the tension is continuous, but for $\delta = 0$, $\dot{\gamma} = 0$, and the tension N is discontinuous. The tension in the $\delta > 0$ case does not, however, approach the discontinuous case as $\delta \to 0$, but rather to a continuous, piecewise linear function of s for each time t. This is because for the folded string tension is negative in the free end, but for the falling string, when the end is released tension vanishes in the falling end. Figure 6 shows the tension in the folded case and in the small δ case. We see that, as $\delta \to 0$, the spatial derivative of tension at the end of the string $N_s(l,t)$ approaches infinity as the end approaches the fold and the speed of the end of the string approaches infinity.

In order to describe the relationship between the initial shape and the number of links and the falling time of the chain, we define the fall time t_f as the first time t that the end is no longer falling downward,

$$t_f = \min\{t | \dot{y}(l,t) > 0\}. \tag{23}$$

In Fig. 7 we see the relationship between the fall time of a chain with initial condition in the shape of a catenary, or a string discretized with n steps, and the number of links in the chain. As the number of links in the chain increases, the behavior is more like an idealized string. Another question is how does the string behave for different values of δ, i.e. different lengths at which the ends are held apart before one end is dropped. In Fig. 8 we see the relation between δ and the

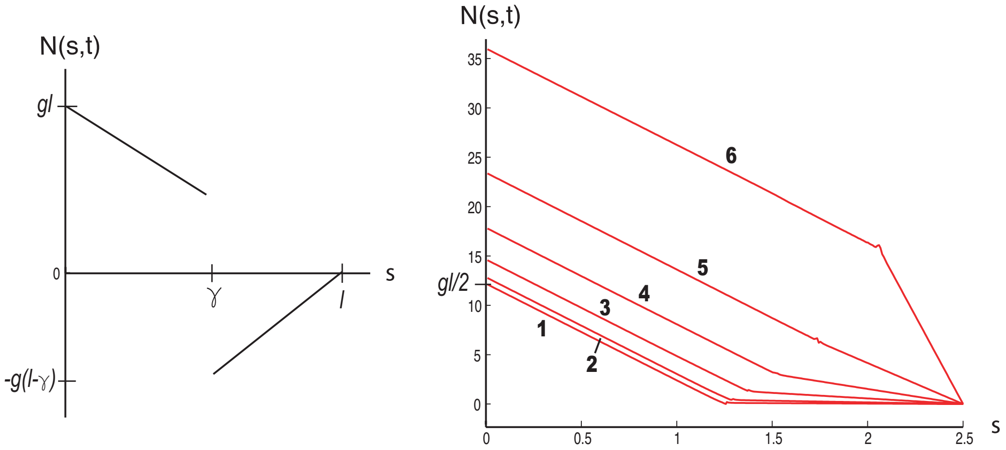


Figure 6: Left panel: Tension in the folded string ($\delta = 0, \dot{\gamma} = 0$). Right panel: Evolution of tension in the falling string for small $\delta > 0$. Number of links is 301; initial condition is a catenary with $\delta = 0.1$. Times are equally spaced: **1** : $t = 0$; **2** : $t = 0.11$; **3** : $t = 0.22$; **4** : $t = 0.32$; **5** : $t = 0.43$; **6** : $t = 0.54$. The time for the string to reach its full extension is $t_f = 0.6$.

fall time of the end of the string. The fall time for the chain does approach the fall time for the idealized folded string limit as $\delta \to 0$, (Fig. 8) but not monotonically. The shortest fall time is around $\delta = 0.5$, and is shorter than the fall time for the idealized folded string. For this initial condition, the end of the chain falls *faster* than the idealized folded string limit, as is seen in (h) of Fig. 9. As $\delta \to 1$, as the initial condition approaches that of a string held horizontal, the fall time approaches that of a falling ball. For $\delta = 1$ the end of the string is governed by $\ddot{y}(l,t) = -g$. (The x coordinate varies, and the behavior of y for $0 < s < l$ is subtle, but the y coordinate of the end of the string evolves like a free falling object under the force of gravity.) Figure 11 shows the evolution of a string with an initial condition of $\delta = 1$. Figure 9 shows the behavior of several different chains.

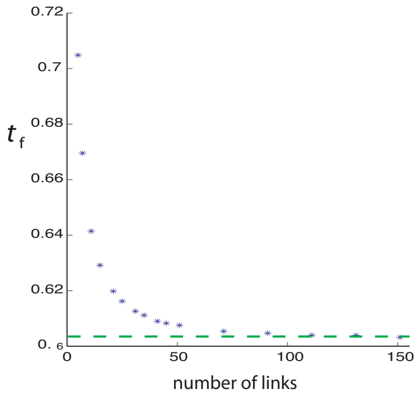


Figure 7: Fall time of end of chain vs. the number of links in a chain. Chain is 2.5 m long. Initial condition is a catenary with $\delta = 0.1$ The dash-dot line is the fall time for the idealized folded string limit (4). The fall time approaches the fall time of the end of a compound pendulum, $0.764s$, as the number of links approaches 1, since a chain with one link is a compound pendulum.

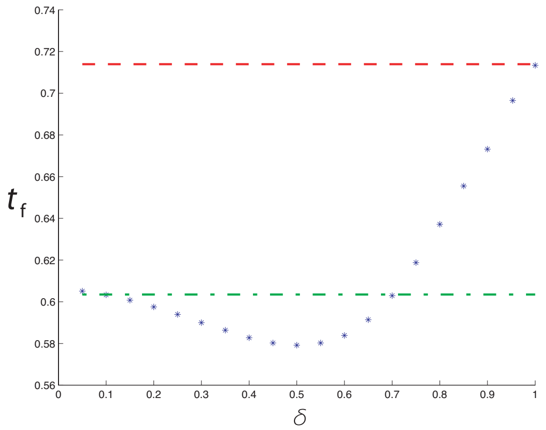


Figure 8: Fall time for the end of chain, for various initial configurations. The chain used here is 2.5 m long, with 201 links. The dashed curve is the time for a free falling object to fall 2.5 m. The dash-dot curve is the fall time for the idealized folded string limit (4).

2.5 The similarity solution–how the string falls

We now investigate how the string falls. With an initial condition in the shape of a catenary, the string "unfolds" in a regular fashion before the tip reaches the fold. This behavior is not like a travelling wave, since the speed constantly increases, and the string does not simply wrap around the shape of the initial condition. Rather, the end that is not falling remains relatively static as the falling end unfolds downward. This suggests a *similarity solution* in a variable $\xi = st^{\alpha}$, for some value of α. With this variable, the first component of equation (7) becomes an ODE in $x(\xi)$ with the balance $\alpha = -2$, so we define the variable

$$\xi = \frac{s}{t^2}. \tag{24}$$

We can use the ansatz of a similarity solution in this variable to explore the behavior of the falling string. Due to the inextensibility condition (9), x_s and y_s can be written in terms of the angle φ as $x_s = \cos(\varphi)$, $y_s = \sin(\varphi)$. Then, the string equations (7) can be written, after differentiating with respect to s, as

$$\begin{aligned} \cos(\varphi)_{tt} &= (N\cos(\varphi))_{ss}, & (25) \\ \sin(\varphi)_{tt} &= (N\sin(\varphi))_{ss}. & (26) \end{aligned}$$

Numerical studies suggest that the tension at each time t is approximately linear in s away from the fold, and that the tension at the left (fixed) end increases approximately proportional to t^2 (Fig. 6). Thus, we set $N(s,t) = t^2 n(\xi)$. Making this substitution, mulitplying the equations by $\cos(\varphi)$ and $\sin(\varphi)$, and adding them together, transforms (25-26) to the following coupled system of nonlinear ODE's:

$$\begin{aligned} n'' - (n - 4\xi^2)(\varphi')^2 &= 0, & (27) \\ (2n' - 6\xi)\varphi' + (n - 4\xi^2)\varphi'' &= 0, & (28) \end{aligned}$$

where the prime denotes differentiation with respect to ξ. The catenary is written as

$$\varphi = \tan^{-1}\left[\mu\left(\frac{s}{l} - \frac{1}{2}\right)\right], \tag{29}$$

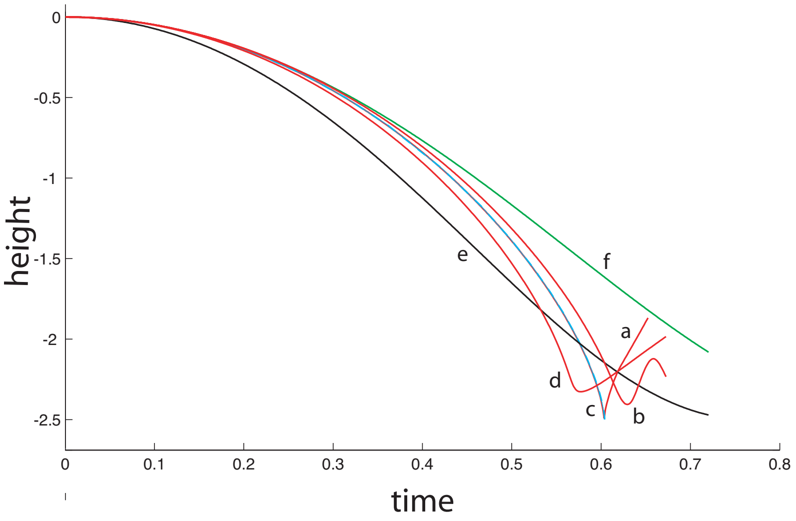


Figure 9: The position of the ends of falling chains, simple pendulum and compound pendulum. (a) A chain with 201 links, $\delta = 0.1$, (b) 15 links, $\delta = 0.1$, (c) idealized folded string limit according to equation (3), (d) chain with 201 links, $\delta = 0.5$, (e) compound pendulum, (f) simple pendulum. Note that (a), the falling chain with δ small, is indistinguishable from the idealized folded string limit (c) before the point of full extension. The fastest falling chain is (d), with $\delta = 0.5$.

where μ is a function of δ. We thus try a similarity solution by substituting ξ for s in (29). This solution does not satisfy the condition that the falling end remains free, $\varphi_s(s = l) = 0$, but it is close for small δ. With this ansatz, n can be calculated by integrating (27-28). In Fig. 10 we compare the numerical solution of the string equations with an initial condition given by (29), with $\delta = 0.1$, with the 'similarity solution' $\varphi = \psi$, where

$$\psi = \tan^{-1}\left[\mu\left(\frac{s}{l\,(1+2.4t^2)} - \frac{1}{2}\right)\right]. \tag{30}$$

The above is not a solution, but in Fig. 10 we see that the falling string unfolds qualitatively like it. We see that the ansatz (30) gives us a qualitative description of how the string falls, before the end of the string reaches its fullest extension.

2.6 The simple pendulum and the string

The simple pendulum cannot be considered to be a point mass on the end of an inextensible string [2], because when the string is slack, the mass is not governed by the pendulum equation. The necessity for a pendulum to consist of a stiff rod is seen in Fig. 11. At time 0 the mass is released, and instantaneously the string ceases to be slack. The end of the string falls at the same speed as a point mass under gravity. A force wave travels down the string, pulling the string to the left. The string only ceases to be slack when the string reaches its full extension. In fact, the point on the end of the string falls at the same speed as a ball, *faster* than the end of a simple pendulum of the same length (Fig. 9). This is due to the fact that for the falling string, while the string is slack, the only force in the y-direction acting on the end of the string is the force of gravity, while in the case of the pendulum, the y component of the force is not equal to mg at the point mass.

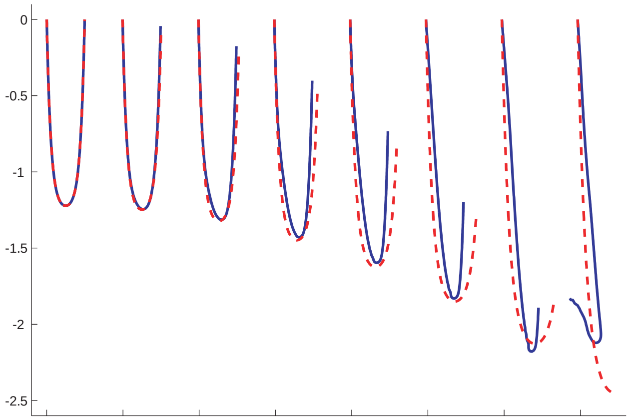


Figure 10: The 'similarity solution' (30) (dashed curve) and the numerical solution of the string equations (solid curve) at times $t = 0, 0.09, 0.18, 0.29, 0.36, 0.45, 0.54, 0.63$.

3 The elastic rod–effects of stiffness

Next we consider the effect of stiffness on a falling rod. We consider a rod that is inextensible and unshearable, and that obeys a linear constitutive relation. Furthermore, we will consider an isotropic rod, *i.e.* we assume that the properties of the material, such as the density and elastic properties, are constant. The rod will be considered to have a circular cross-section with constant radius R. The assumption of circular cross-sections is not necessary, but it makes the calculation of the stiffness simpler.

A general formulation for the dynamics of planar rods is given in [1]. Since the rod is inextensible, $|\mathbf{r_s}| = \mathbf{1}$, so we can write the equations in terms of the angle φ (see Fig. 12) between the x-axis and the tangent to the curve $\mathbf{t} = \mathbf{r}_s$,

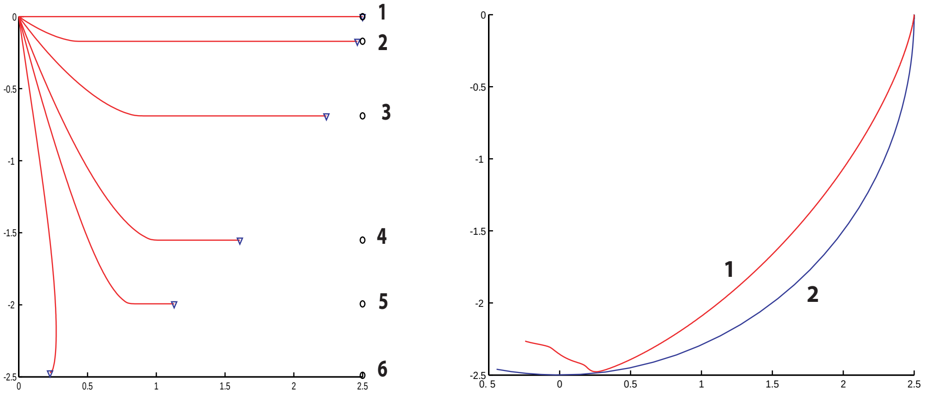
$$x_s = \cos(\varphi), \tag{31}$$
$$y_s = \sin(\varphi). \tag{32}$$

Balancing the linear and angular momentum across cross-sections gives

$$\rho A\,\ddot{x} = F_s, \tag{33}$$
$$\rho A\,\ddot{y} = G_s - \rho A g, \tag{34}$$
$$\rho I\,\ddot{\varphi} = EI\,\varphi_{ss} + G\cos(\varphi) - F\sin(\varphi). \tag{35}$$

where ρ is the density per unit volume, A is the area of the cross-section, E is the material Young's modulus, and I is the moment of inertia of the cross-section. Differentiating (33-34) with respect to s, and using (31-32) yields a closed system of equations. Note that the equations for the rod (33-35) reduce to the equations for the string (6, 7) when $I = 0$, i.e. when the moment of inertia of the cross-section is zero. As I tends to zero the rod "tends to" the string. However, note that the string equations are obtained by setting $I = 0$, but taking $\rho A > 0$. That is, a string is a rod with a positive mass per unit length but zero moment of inertia. Since $A = \pi R^2$ and $I = \pi R^4/4$, this cannot be obtained with a solid elastic material, which is why a string should be conceived of as a chain with infinitely many links.

The evolution of solutions depends primarily on the boundary conditions. For a falling rod the end that falls is *free*, that is, no force or moment is imposed. The boundary condition at this end is that the tension and curvature vanish, $\varphi_s = F = G = 0$. We take the left end to be fixed, which



(a)

(b)

Figure 11: (a) Chain with 301 links and ball at times (in seconds): **1**: 0, **2**: 0.1872, **3**: 0.3748, **4**: 0.5624, **5**: 0.6374, **6**: 0.7125. (b) Curve 1 is the path travelled by a point mass on the end of a string (the triangles in Fig. (a)) in 0.75 seconds; curve 2 is the path travelled by the end of a simple pendulum in 1 second.

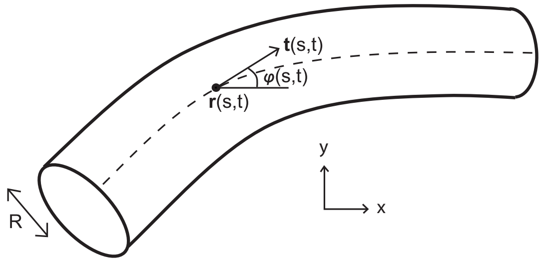


Figure 12: Elastic rod in the plane.

determines F_s and G_s at $s = 0$. For φ at $s = 0$, if the rod is allowed to freely swivel at its point of attachment, then $\varphi_s = 0$ at $s = 0$.

$$\varphi_s(0,t) = 0,\ F_s(0,t) = 0,\ G_s(0,t) = \rho A g, \tag{36}$$

$$\varphi_s(l,t) = 0,\ F(l,t) = G(l,t) = 0. \tag{37}$$

The system (33 - 35) has an associated energy,

$$\mathcal{H} = \frac{1}{2}\int_0^1 \left[\rho I \dot{\varphi}^2 + \rho A\left(\dot{x}^2 + \dot{y}^2\right) + EI\varphi_s^2 + 2\rho A y\right] ds. \tag{38}$$

A simple calculation shows that for a smooth rod with the boundary conditions (36-37) energy is conserved, $d\mathcal{H}/d\tilde{t} = 0$.

Investigations of the behavior of the falling rod are carried out using a numerical finite difference scheme developed in [4], adapted to account for the force of gravity. Conservation of energy is used to check the accuracy of solutions. Two simulations are shown in Fig. 13. In Fig. 14 we see the relationship between the stiffness and the fall time. We take as the initial condition a catenary, with δ set to 0.5. R is a measure of the stiffness of the rod, which is $EI = E\pi R^4/4$, and the length of the rod is held constant. The material properties used here are those of rubber, $E = 1.7 \cdot 10^6 N/m^3$, $\rho = 1100\ kg/m^3$, but the results are qualitatively similar for different properties of the material;

the parameter which has a qualitative effect on behavior is λ, a measure of the inverse of stiffness. We see that for relatively large values of the stiffness, a decrease in stiffness *decreases* the fall time. This is due to the fact that to obtain an initial condition in the shape of a catenary the rod must be bent, and thus an elastic energy is stored in the rod. When one end is released, the elastic energy is transferred into downward movement of the end of the rod, the elastic energy contributes to the potential energy of gravity in creating downward movement. For low values of the stiffness a decrease in the stiffness increases the fall time. In this case, the stiffness has a dampening effect. There is thus a critical value, around $R = .025$ in Fig. 14, of the stiffness at which the dampening effect of the stiffness overcomes the tendency of springing downward.

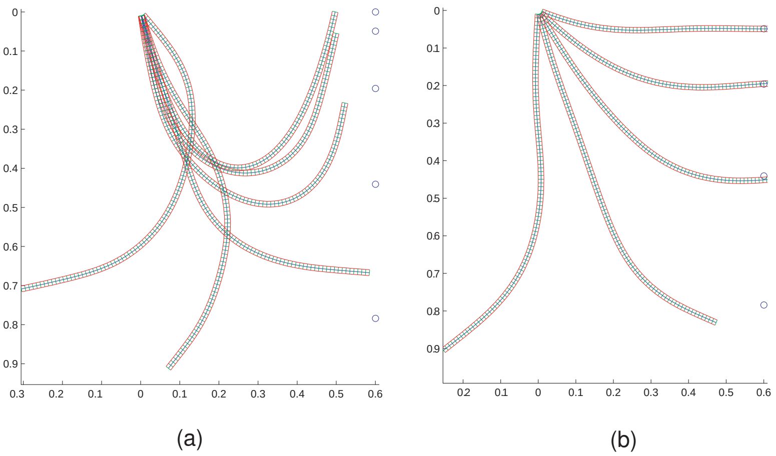


Figure 13: Falling rods for (a) $\delta = 0.5$ and (b) $\delta = 1$, at times $t = 0.1, 0.2, 0.3, 0.4, 0.5s$. The circles are the positions of a free falling ball at the same times. (b) is the “flexible compound pendulum.” The greater the stiffness, the closer the behavior of the rod is to the compound pendulum when $\delta = 1$.

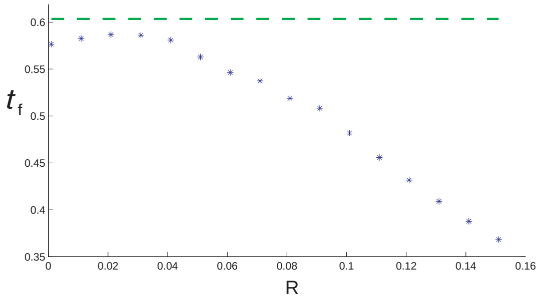


Figure 14: Effect of stiffness. Fall time of end of rod vs. radius of rod. R is proportional to (the fourth root of) bending stiffness. The horizontal dashed line is the fall time of the idealized folded string limit (4). The rod is 2.5 m long; the initial condition is a catenary with $\delta = 0.5$. Notice that in all cases the rod falls faster than an the limiting idealized folded string.

4 Discussion

Schagerl, et.al. [8] performed a beautiful series of experiments showing that the end of the falling chain falls faster than a free falling object. They showed experimentally and numerically what happens to a string when it is very close to the idealized folded string configuration. Yet, contradictory results regarding this limit remained unexplained. We have shown the flaws in earlier treatments, and thus reconciled these contradictory results. The traditional analysis culminating in equation (3) is correct, but only in a limiting sense. Previous contradictory results have been derived for the degenerate case when the derivative of the space curve are discontinuous. O'Reilly and Varadi [5] consider the falling string as a particular application of a broader theory of shocks in thermomechanical media. In light of the results of the present paper, the results in [5] should be reconsidered.

We have also shown that the relationship between the fall time and the parameters of stiffness, number of links in a chain and the distance the ends are held apart before releasing one end, do not follow simple rules, and are not monotonic. Perhaps the most interesting result is that the end of the string falls fastest when the distance between the ends prior to release is approximately 1/2 the length of the string.

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